

Complex Analysis 第三章习题¹

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CHAPTER 3

13. Suppose $f(z)$ is holomorphic in a punctured disc $D_r^*(z_0)$. Suppose also that

$$|f(z)| \leq |z - z_0|^{-1+\epsilon} \quad (3.1)$$

for some $\epsilon > 0$ and all z near z_0 . Show that the singularity of f at z_0 is removable.

证明. Since

$$\lim_{z \rightarrow z_0} |(z - z_0)f(z)| = 0 \quad (3.2)$$

Then by 可去奇点的等价刻画 (笔记 **Cor 8.3.2**),

z_0 is a removable singularity of f . □

14. Prove that all entire functions that are also injective take the form

$$f(z) = az + b \text{ with } a, b \in \mathbb{C}, a \neq 0 \quad (3.3)$$

证明. Let $F : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$, $F(z) = f(\frac{1}{z})$.

Since f entire, then f has an isolated singularity at ∞ , so does F at $z = 0$.

(a) f has no removable singularity at ∞ .

(Otherwise f is constant, contradiction to f is injective.)

(b) f has no essential singularity at ∞ .

反证法. Assume f has an essential singularity at ∞ , so does F at $z = 0$.

$\forall |z_0| > r > 0$, $|z_0| - r > r' > 0$, then

$$D_r(z_0) \cap D_{r'}^*(0) = \emptyset \quad (3.4)$$

Since F is holomorphic and nonconstant in $D_r(z_0)$, then by **Open Mapping Theorem**,

$$\begin{aligned} F \text{ is open in } D_r(z_0). \quad F(D_r(z_0)) \underset{\text{open}}{\subset} \mathbb{C} \\ \Rightarrow \exists r_0 > 0, D_{r_0}(F(z_0)) \subset F(D_r(z_0)) \end{aligned}$$

Since F has an essential singularity at $z = 0$, then by **Casorati-Weierstrass Theorem**,

$F(D_{r'}^*(0))$ is dense in $\mathbb{C}$, while $D_{r_0}(F(z_0))$ open.

$$\Rightarrow D_{r_0}(F(z_0)) \cap F(D_{r'}^*(0)) \neq \emptyset \quad (3.5)$$

$$\Rightarrow F(D_{r'}^*(0)) \cap F(D_r(z_0)) \neq \emptyset \quad (3.6)$$

which is a contradiction with F injection and $D_r(z_0) \cap D_{r'}^*(0) = \emptyset$.

(c) Therefore, f has a pole of order n at ∞ , i.e.

$$f(z) = \sum_{k=0}^n a_k z^k, \quad a_n \neq 0 \quad (3.7)$$

If $n > 1$, then

Either $f(z) = 0$ or $f(z) = 1$ has more than 1 roots, contradiction to f is injective.

Therefore, $n = 1$, $f(z) = az + b$.

□

15. Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

(a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B \quad (3.8)$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

证明. By **Cauchy's Inequality**,

$$|f^{(n)}(0)| \leq \frac{n! \|f\|_{C_R(0)}}{R^n} \leq \frac{n!(AR^k + B)}{R^n}, \quad \forall R > 0 \quad (3.9)$$

Then for $\forall n > k$, letting $R \rightarrow \infty$, we get

$$f^{(n)}(0) = 0, \quad \forall n > k \quad (3.10)$$

Since f entire, then $\exists r > 0$, s. t.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^k a_n z^n, \quad \forall z \in D_r(0), \quad a_n = \frac{f^{(n)}(0)}{n!} \quad (3.11)$$

Then $f^{(k+1)}(z) = 0, \quad \forall z \in D_r(0)$. Thus $f^{(k+1)} = 0, \quad \forall z \in \mathbb{C}$.

Therefore, f polynomial of degree $\leq k$. □

- (c) Let $w_1, \dots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points w_j , $1 \leq j \leq n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points w_j , $1 \leq j \leq n$, is exactly equal to 1.

证明. Let $f(z) = (z - w_1) \cdots (z - w_n)$, then f is holomorphic on $\overline{\mathbb{D}}$.

Since $f(0) = |w_1| \cdots |w_n| = 1$, then by **Maximum Modulus Principle**,

$$\sup_{z \in \partial \mathbb{D}} |f(z)| \geq |f(0)| = 1 \quad (3.12)$$

Therefore $\exists z \in \partial \mathbb{D}$, s. t. $|f(z)| \geq 1$.

Let $g(\vartheta) = |f(e^{i\vartheta})|$, $\vartheta \in [0, 2\pi)$. Then g is continuous.

Suppose $w_1 = e^{i\vartheta_1} \in \partial \mathbb{D}$, $z = e^{i\vartheta_2} \in \partial \mathbb{D}$, 不妨设 $\vartheta_1 < \vartheta_2$.

Since

$$g(\vartheta_1) = |f(w_1)| = 0 < 1 \leq |f(z)| = g(\vartheta_2) \quad (3.13)$$

Then $\exists \vartheta_0 \in (\vartheta_1, \vartheta_2)$, s. t.

$$g(\vartheta_0) = |f(e^{i\vartheta_0})| = 1 \quad (3.14)$$

Let $w = e^{i\vartheta_0} \in \partial \mathbb{D}$. □

(d) Show that if the real part of an entire function f is bounded, then f is constant.

证明. Let $F : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$, $F(z) = f(\frac{1}{z})$.

Since f is entire, then f has an isolated singularity at ∞ , so does F at $z = 0$.

- f has no pole at ∞ .

Otherwise, assume f has a pole of order n at ∞ , then f is a nonconstant polynomial

$$f(z) = \sum_{k=0}^n a_k z^k \quad (3.15)$$

which is a contradiction to $Re(f)$ is bounded.

- f has no essential singularity at ∞ .

Otherwise, by **Casorati-Weierstrass Theorem**, $\exists r > 0$, s. t.

$$F(D_r^*(0)) \text{ is dense in } \mathbb{C}$$

$$\Rightarrow Re(f) \text{ is not bounded, contradiction.}$$

综上, f has a removable singularity at ∞ , which means f is constant. □

16. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z) \quad (3.16)$$

Show that if ϵ is sufficiently small, then

- (a) $f_\epsilon(z)$ has a unique zero in $|z| \leq 1$, and
- (b) if z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

证明.

- (a) Since $\overline{\mathbb{D}}$ is compact, $|g|$ continuous on $\overline{\mathbb{D}}$, then $|g|$ attains its maximum on $\overline{\mathbb{D}}$. i.e.
 $\exists M > 0$, s. t.

$$|g| \leq M, \quad \forall z \in \overline{\mathbb{D}} \quad (3.17)$$

Since $\partial\mathbb{D}$ is also compact, $|f|$ continuous on $\partial\mathbb{D}$ and $|f| > 0, \forall z \in \partial\mathbb{D}$, then $|f|$ attains its minimum $m \geq 0$, i.e.

$$|f| \geq m > 0, \quad \forall z \in \partial\mathbb{D} \quad (3.18)$$

Let $\epsilon < \frac{m}{M}$, then $|f_\epsilon| = |f + \epsilon g| > |\epsilon g| - |f|$, where $|\epsilon g| - |f| < 0, \forall z \in \partial\mathbb{D}$.

Thus

$$|f| + |f_\epsilon| > |f| + |\epsilon g| - |f| = |\epsilon g| = |f - f_\epsilon|, \quad \forall z \in \partial\mathbb{D} \quad (3.19)$$

By **Rouché's Theorem**,

f and f_ϵ have the same number of zeros in $\mathbb{D}$, which is exactly 1.

Since $|f_\epsilon| = |f + \epsilon g| \geq |f| - |\epsilon g| > 0, \forall z \in \partial\mathbb{D}$, f_ϵ has no zero on $\partial\mathbb{D}$

Therefore, f_ϵ has exactly 1 zero on $\overline{\mathbb{D}}$, which is actually in $\mathbb{D}$.

(b) Let $h(\epsilon) = z_\epsilon$. Fix ϵ and z_ϵ . We denote

$$N_r := D_r(z_\epsilon), \quad \forall r > 0 \quad (3.20)$$

Since $\overline{\mathbb{D}} \setminus N_r, \quad \forall r > 0$ is compact, and $|f_\epsilon|$ continuous on $\overline{\mathbb{D}} \setminus N_r$

Then for $r > 0, \exists \delta_r > 0$, s. t.

$$|f_\epsilon| \geq \delta_r > 0, \quad \forall z \in \overline{\mathbb{D}} \setminus N_r, \quad \forall r > 0 \quad (3.21)$$

$\forall |\epsilon' - \epsilon| < \frac{\delta_r}{M}$, we have

$$|f_{\epsilon'} - f_\epsilon| = |\epsilon' - \epsilon| \cdot |g| < \delta_r, \quad \forall z \in \overline{\mathbb{D}} \quad (3.22)$$

Thus

$$f_{\epsilon'}(z) \neq 0, \quad \forall z \in \overline{\mathbb{D}} \setminus N_r$$

(Otherwise if $\exists z_0 \in \overline{\mathbb{D}} \setminus N_r$, s. t. $f_{\epsilon'}(z_0) = 0$, then $|f_\epsilon(z_0)| < \delta_r$,

这与 $|f_\epsilon| \geq \delta_r, \quad \forall z \in \overline{\mathbb{D}} \setminus N_r$ 矛盾)

Since $f_{\epsilon'}(z) \neq 0, \quad \forall z \in \overline{\mathbb{D}} \setminus N_r$, we can get

$$z_{\epsilon'} \in N_r, \quad |z_{\epsilon'} - z_\epsilon| < r \quad (3.23)$$

Therefore, $\forall r > 0, \exists \delta_r > 0$, s. t.

$$|z_{\epsilon'} - z_\epsilon| < r, \quad \forall |\epsilon' - \epsilon| < \frac{\delta_r}{M} \quad (3.24)$$

i.e.

$$|h(\epsilon') - h(\epsilon)| < r, \quad \forall |\epsilon' - \epsilon| < \frac{\delta_r}{M} \quad (3.25)$$

which means h is continuous.

□

17. Let f be non-constant and holomorphic in an open set containing the closed unit disc.

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

(b) If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

证明.

(a) $\forall |w_0| < 1$. Since $|f| = 1 > |w_0|$, $\forall z \in \partial\mathbb{D}$, then by **Rouché's Theorem**,

f and $f - w_0$ have the same number of zeros in $\mathbb{D}$, $\forall |w_0| < 1$.

$\Rightarrow$ If f has a zero in $\mathbb{D}$, then $\mathbb{D} \subset \text{Image}(f)$.

下面证明 f has zeros in $\mathbb{D}$:

反证法. Assume f is nonvanishing on $\mathbb{D}$, then

$\frac{1}{f}$ is nonconstant and holomorphic on $\mathbb{D}$, with $\left|\frac{1}{f}\right| = 1$, $\forall z \in \partial\mathbb{D}$

By **Maximum Modulus Principle**,

$$\left|\frac{1}{f}\right| \leq 1, \quad \forall z \in \overline{\mathbb{D}} \quad (3.26)$$

Then

$$|f| \geq 1, \quad \forall z \in \overline{\mathbb{D}} \quad (3.27)$$

But by **Maximum Modulus Principle**,

$$|f| \leq 1, \quad \forall z \in \overline{\mathbb{D}} \quad (3.28)$$

Therefore, $|f| \equiv 1$, $\forall z \in \overline{\mathbb{D}}$. Since f is holomorphic on $\overline{\mathbb{D}}$, then f constant, 矛盾.

Therefore, f has zeros in $\mathbb{D} \Rightarrow \mathbb{D} \subset \text{Image}(f)$

(b) 同理, $\forall |w| < 1$. Since $|f| \geq 1 > |w|$, $\forall z \in \partial\mathbb{D}$, then by **Rouché's Theorem**,

f and $f - w$ have the same number of zeros in $\mathbb{D}$, $\forall |w| < 1$.

Since $\exists w_0 = z_0 \in \mathbb{D}$, s. t.

$$f(w_0) - w_0 = 0 \tag{3.29}$$

Then $f(z) = 0$ has at least one root in $\mathbb{D}$, so does $f(z) - w = 0$, $\forall |w| < 1$.

$\Rightarrow \mathbb{D} \subset \text{Image}(f)$

□

19. Prove the maximum principle for harmonic functions, that is:

- (a) If u is a non-constant real-valued harmonic function in a region Ω , then u cannot attain a maximum (or a minimum) in Ω .
- (b) Suppose that Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)| \quad (3.30)$$

证明.

- (a) 反证法. Suppose u attains its maximum at $z_0 \in \Omega$, then let f be a holomorphic function in $D_r(z_0)$ for some $r > 0$, s. t. $\operatorname{Re} f = u$, and

$$f(z_0) = u(z_0) + iv(z_0) \quad (3.31)$$

Since u is nonconstant, so is f . By **Open Mapping Theorem**

f is open in $D_r(z_0)$, $f(D_r(z_0))$ open.

Thus $\exists \delta > 0$, s. t.

$$D_\delta(f(z_0)) \subset f(D_r(z_0)) \quad (3.32)$$

Since u attains its maximum at z_0 , $u(z) \leq u(z_0)$, $\forall z \in \Omega$. Then

$$\forall c = u(z_0) + \frac{\delta}{3} + iv(z_0) \in D_\delta(f(z_0)),$$

$$c \notin f(D_r(z_0)), \text{ contradiction to } D_\delta(f(z_0)) \subset f(D_r(z_0)) \quad (3.33)$$

Therefore, u cannot attain its maximum in Ω .

(b) Since $\overline{\Omega}$ is compact, $|u|$ continuous on $\overline{\Omega}$, then $|u|$ attains its maximum on $\overline{\Omega}$.

Suppose $|u|$ nonconstant, then by (a),

$|u|$ cannot attain its maximum in Ω .

Therefore

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)| \quad (3.34)$$

□

22. Show that there is no holomorphic function f in the unit disc $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ such that

$$f(z) = \frac{1}{z}, \quad \forall z \in \partial\mathbb{D} \quad (3.35)$$

证明. 反证法. Assume f satisfies the conditions.

Since $\overline{\mathbb{D}}$ is compact, f continuous on $\overline{\mathbb{D}}$, then f uniformly continuous on $\overline{\mathbb{D}}$.

i.e. $\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \epsilon, \quad z_1, z_2 \in \overline{\mathbb{D}} \quad (3.36)$$

Then

$$\left| \int_{C_1(0)} \left(f(z) - f\left(\left(1 - \frac{\delta}{2}\right)z\right) \right) dz \right| \leq 2\pi\epsilon \quad (3.37)$$

$$\Rightarrow \left| \int_{C_1(0)} f(z) dz - \frac{1}{1 - \frac{\delta}{2}} \int_{C_{1-\frac{\delta}{2}}(0)} f(z) dz \right| \leq 2\pi\epsilon \quad (3.38)$$

By **Cauchy's Theorem**,

$$\int_{C_{1-\frac{\delta}{2}}(0)} f(z) dz = 0 \quad (3.39)$$

Then

$$\left| \int_{C_1(0)} f(z) dz \right| \leq 2\pi\epsilon, \quad \forall \epsilon > 0 \quad (3.40)$$

Letting $\epsilon \rightarrow 0^+$, we get

$$\int_{C_1(0)} f(z) dz = 0 \quad (3.41)$$

which is a contradiction to

$$\int_{C_1(0)} f(z) dz = \int_{C_1(0)} \frac{1}{z} dz = 2\pi i \neq 0 \quad (3.42)$$

□