

Complex Analysis 第八章习题¹

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¹课堂教材：《*Complex Analysis*》— *Elias M. Stein*

CHAPTER 8

1. A holomorphic mapping $f : U \rightarrow V$ is a local bijection on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z , so that $f : D \rightarrow f(D)$ is a bijection. Prove that a holomorphic map $f : U \rightarrow V$ is a local bijection on U if and only if $f'(z) \neq 0$ for all $z \in U$.

证明.

- (a) 必要性 $\Rightarrow$: $\forall z_0 \in U, \exists$ open disc $D \subset U$ centered by z_0 , s. t.

$$f : U \rightarrow f(D) \text{ is a bijection}$$

Then $f|_D : D \rightarrow f(D)$ is holomorphic and bijective, particularly injective.

By **Prop 12.2.1**(笔记), $f'(z) \neq 0, \forall z \in D$, specifically

$$f'(z_0) \neq 0, \forall z_0 \in U \quad (8.1)$$

Therefore, $f'(z) \neq 0, \forall z \in U$.

- (b) 充分性 $\Leftarrow$: $\forall z_0 \in U$, 记 $w_0 = f(z_0)$.

Since $f'(z_0) \neq 0$, then $\exists r > 0$, s. t. $f(z) \neq w_0, \forall z \in D_r^*(z_0)$. Let

$$m = \min \left\{ |f(z) - w_0| \mid |z - z_0| = \frac{R}{2} \right\} \quad (8.2)$$

Since $C_{\frac{R}{2}}(z_0)$ is compact, then m 可被取到, 且 $m > 0$.

Since $\forall |w - w_0| < m$,

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < m \leq |f(z) - w_0|, \forall z \in C_{\frac{R}{2}}(z_0) \quad (8.3)$$

Then by **Rouché's Theorem**, for $\forall |w - w_0| < m$,

$f(z) - w$ and $f(z) - w_0$ have the same number of zeros in $D_{\frac{R}{2}}(z_0)$, which is exactly 1.

Since f is continuous, then $\exists \delta > 0$, s. t.

$$|f(z) - w_0| < m, \quad \forall z \in D_\delta(z_0) \quad (8.4)$$

Let $r = \min \{\frac{R}{2}, \delta\} > 0$, then

$$|f(z) - w_0| < m, \quad \forall z \in D_r(z_0) \quad (8.5)$$

且 $f(z) = w$ 至多含有一个零点, $\forall |w - w_0| < m$.

$\Rightarrow f : D_r(z_0) \rightarrow f(D_r(z_0))$ is bijective.

□

2. Suppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0 , are orthogonal at z_0 , and so that F restricted to Γ_1 is real and has a minimum at z_0 , while F restricted to Γ_2 is also real but has a maximum at z_0 .

证明. 下面主要分为两步进行证明.

- (a) $\exists g$ holomorphic, s. t. $F(z) = (g(z))^2$ in some neighbourhood of z_0 :

Since z_0 is a zero of order 2 of F , then $\exists$ holomorphic and nonvanishing $h : D_r(z_0) \rightarrow \mathbb{C}$, s. t.

$$F(z) = (z - z_0)^2 h(z), \quad \forall z \in D_r(z_0) \quad (8.6)$$

Since h is holomorphic and nonvanishing on $D_r(z_0)$, then $\exists \varphi$ holomorphic, s. t.

$$h(z) = (\varphi(z))^2, \quad \forall z \in D_r(z_0) \quad (8.7)$$

Take $g(z) = (z - z_0)\varphi(z)$, then g is holomorphic and

$$F(z) = (g(z))^2, \quad \forall z \in D_r(z_0) \quad (8.8)$$

- (b) Construct Γ_1, Γ_2 :

Since $F(z) = (g(z))^2$ and $F(z_0) = F'(z_0) = 0, F''(z_0) \neq 0$, then we can get

$$\begin{cases} g(z_0) = 0 \\ g'(z_0) \neq 0 \end{cases} \quad (8.9)$$

Then by **Exercise 1**, $\exists \delta > 0$, s. t. $g : D_\delta(z_0) \rightarrow g(D_\delta(z_0))$ conformal.

Thus $g^{-1} : g(D_\delta(z_0)) \rightarrow D_\delta$ is also conformal.

Then we define the curves by

$$\begin{cases} \gamma_1(t) = t, & t \geq 0 \\ \gamma_2(t) = it, & t \geq 0 \end{cases}, \text{ and } \begin{cases} \Gamma_1(t) = g^{-1}(\gamma_1(t)), & t \geq 0 \\ \Gamma_2(t) = g^{-1}(\gamma_2(t)), & t \geq 0 \end{cases} \quad (8.10)$$

Therefore,

$$\begin{cases} F(\Gamma_1(t)) = t^2, & t \geq 0 \\ F(\Gamma_2(t)) = -t^2, & t \geq 0 \end{cases} \quad (8.11)$$

□

3. Suppose U and V are conformally equivalent. Prove that if U is simply connected, then so is V . Note that this conclusion remains valid if we merely assume that there exists a continuous bijection between U and V .

证明. U and V conformally equivalent $\Rightarrow$ 同胚 $\Rightarrow$ 基本群的拓扑不变性.

□

4. Does there exist a holomorphic surjection from the unit disc to $\mathbb{C}$?

解. 存在. 下面进行构造:

Since $F(z) = \frac{-z+i}{z+i} : \mathbb{H} \rightarrow \mathbb{D}$ conformal, then

$$f(z) = F^{-1}(z) = i \cdot \frac{1-z}{1+z} : \mathbb{D} \rightarrow \mathbb{H} \text{ conformal}$$

Let $g(z) = z - i$ (将上半平面 $\mathbb{H}$ 向下延伸 i , 使其包含实轴 $\mathbb{R}$) and $h(z) = z^2$. 记

$$\mathbb{H}' = g(\mathbb{H}), \mathbb{H} \subsetneq \mathbb{H}', \mathbb{R} \subset \mathbb{H}' \quad (8.12)$$

$\forall w \in \mathbb{C}$, 根据代数基本定理, $\exists z_0 \in \mathbb{C}$, s. t. $z_0^2 = w \Rightarrow (-z_0)^2 = w$.

$$\begin{cases} \text{if } z_0 \in \mathbb{R} \Rightarrow z_0 \in \mathbb{H}' \\ \text{if } z_0 \notin \mathbb{R}, \text{ then either } z_0 \text{ or } -z_0 \text{ is in } \mathbb{H}, \text{ thus in } \mathbb{H}' \end{cases} \quad (8.13)$$

Therefore $h(g(\mathbb{H})) = \mathbb{C}$ is surjective.

$\Rightarrow$ Let $\varphi = h \circ g \circ f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and surjective.

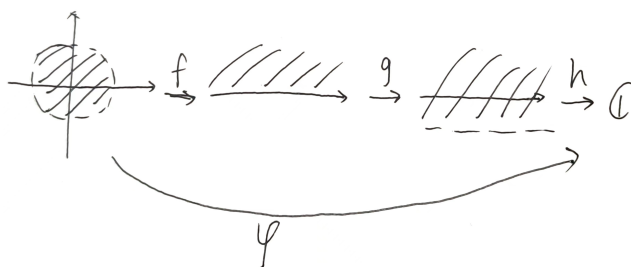


图 8.1: holomorphic & surjective φ

□

13. The pseudo-hyperbolic distance between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right| \quad (8.14)$$

(a) Prove that if $f : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \leq \rho(z, w), \quad \forall z, w \in \mathbb{D} \quad (8.15)$$

Moreover, prove that if f is an automorphism of $\mathbb{D}$ then f preserves the pseudo-hyperbolic distance

$$\rho(f(z), f(w)) = \rho(z, w), \quad \forall z, w \in \mathbb{D} \quad (8.16)$$

证明.

(a) Denote the Blaschke Factor

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad \forall z, a \in \mathbb{D} \quad (8.17)$$

Then $\varphi_a \in \text{Aut}(\mathbb{D})$ and $\varphi_a \circ \varphi_a(z) = z$.

Fix $w \in \mathbb{D}$. Let $g = \varphi_{f(w)} \circ f \circ \varphi_w : \mathbb{D} \rightarrow \mathbb{D}$, then g is holomorphic, and

$$g(0) = \varphi_{f(w)} \circ f(w) = \varphi_{f(w)}(f(w)) = \frac{f(w) - f(w)}{1 - |f(w)|^2} = 0 \quad (8.18)$$

Then by **Schwartz Lemma**,

$$|g(z)| \leq |z|, \quad \forall z \in \mathbb{D} \quad (8.19)$$

i.e.

$$|\varphi_{f(w)} \circ f \circ \varphi_w(z)| \leq |z|, \quad \forall z, w \in \mathbb{D} \quad (8.20)$$

Let $z' = \varphi_w(z)$, then $z = \varphi_w(z')$, and

$$|\varphi_{f(w)}(f(z'))| \leq |\varphi_w(z')| \quad (8.21)$$

Since $\varphi_w \in \text{Aut}(\mathbb{D})$, $z \in \mathbb{D}$, then z' 可取遍 $\mathbb{D}$. Thus

$$|\varphi_{f(w)}(f(z))| \leq |\varphi_w(z)|, \quad \forall z, w \in \mathbb{D} \quad (8.22)$$

i.e.

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(z)}f(w)} \right| \leq \left| \frac{z - w}{1 - \bar{z}w} \right|, \quad \forall z, w \in \mathbb{D} \quad (8.23)$$

which proves the prop.

□

13. The pseudo-hyperbolic distance between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right| \quad (8.24)$$

(b) Prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{D} \quad (8.25)$$

This result is called the Schwarz-Pick lemma.

证明.

(b) By (a),

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} \right| \leq \left| \frac{z - w}{1 - \overline{w}z} \right|, \quad \forall z, w \in \mathbb{D} \quad (8.26)$$

Thus

$$\left| \frac{f(z) - f(w)}{z - w} \cdot \frac{1}{1 - \overline{f(w)}f(z)} \right| \leq \frac{1}{|1 - \overline{w}z|}, \quad \forall z, w \in \mathbb{D} \quad (8.27)$$

Fix $z \in \mathbb{D}$. Letting $w \rightarrow z$, we get

$$\left| \frac{f'(z)}{1 - |f(z)|^2} \right| \leq \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{D} \quad (8.28)$$

Since $f : \mathbb{D} \rightarrow \mathbb{D}$, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{D} \quad (8.29)$$

□