

*Complex Analysis*¹

–TW–

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¹课堂教材：《*Complex Analysis*》— *Elias M. Stein*

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第零章 课程要求

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第一章 Week 1

1.1 复数的引入

引入

下面从代数结构 (Group, Ring, Field) 的角度引入复数的概念.

Consider the set $\mathbb{R}^2$. Define two operations. $\forall (a, b), (c, d) \in \mathbb{R}^2$,

$$(a, b) + (c, d) := (a + c, b + d) \quad (1.1)$$

$$(a, b) \cdot (c, d) := (ac - bd, bc + ad) \quad (1.2)$$

" $\cdot$ " is commutative.

" $+$ ", " $\cdot$ " satisfy associative and distributive laws.

$$(0, 0) : \text{The additive identity} \quad (1.3)$$

$$(1, 0) : \text{The multiplicative identity} \quad (1.4)$$

$\Rightarrow (\mathbb{R}^2, +, \cdot)$ is a communicative ring.

$\forall (a, b) \in \mathbb{R}^2, (a, b) \neq (0, 0)$, if

$$(a, b) \cdot (x, y) = (1, 0) \quad (1.5)$$

$$\Rightarrow x = \frac{a}{a^2 + b^2}, y = \frac{-b}{a^2 + b^2} \quad (1.6)$$

Therefore, $(\mathbb{R}^2, +, \cdot)$ is a field, denoted as $\mathbb{C}$.

复数的乘法 在上述对 $\mathbb{C}$ 的定义中, 唯一非平凡的点便是乘法运算 " $\cdot$ " 的定义.

下面我们从代数的方法, 从另一个角度理解复数的乘法.

We may ask a question : Can we define a "·" and let $(\mathbb{R}^3, +, \cdot)$ be a field?

However, the answer is certainly not!

Consider $M_2 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ equipped with the usual matrix addition and multiplication.

Define a map σ .

$$\sigma : \mathbb{R}^2 \longrightarrow M_2 \quad (1.7)$$

$$(a, b) \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad (1.8)$$

Then, σ is bijective.

$$\sigma(a, b) \cdot \sigma(c, d) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix} = \sigma((a, b) \cdot (c, d)) \quad (1.9)$$

$\Rightarrow \sigma$ is an isomorphism(同构映射).

于是复数乘法可视作复平面上带伸缩的旋转.

1.2 复数的基本性质

Some Facts

$$|\operatorname{Re} z| \leq |z|, \quad |\operatorname{Im} z| \leq |z| \quad (1.10)$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i} \quad (1.11)$$

性质 下面给出一些命题.

1. 三角不等式.

命题 1.2.1 (Triangle Inequality). *Let $z, w \in \mathbb{C}$. Then*

$$|z + w| \leq |z| + |w| \quad (1.12)$$

证明. *Let $z = a + bi$, $w = c + di$. Then*

$$\Leftrightarrow \sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \quad (1.13)$$

$$\Leftrightarrow ac + bd \leq \sqrt{(a^2 + b^2)(c^2 + d^2)} = \sqrt{(ac)^2 + (bd)^2 + a^2 d^2 + b^2 c^2} \quad (1.14)$$

□

推论 1.2.1. *If $z, w \in \mathbb{C}$, then*

$$||z| - |w|| \leq |z - w| \quad (1.15)$$

证明.

$$|z| = |z - w + w| \leq |z - w| + |w| \quad (1.16)$$

$$|w| = |z - w - z| \leq |z - w| + |z| \quad (1.17)$$

$$\Rightarrow |z - w| \geq \max\{|z| - |w|, |w| - |z|\} = ||z| - |w|| \quad (1.18)$$

□

2. Cauchy – Schwarz 不等式.

命题 1.2.2 (Cauchy – Schwarz). *Let $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$. Then*

$$\left| \sum_{k=1}^n z_k w_k \right|^2 \leq \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) \quad (1.19)$$

证明. $\forall \hat{\eta} \in \mathbb{R}, \vartheta \in \mathbb{R},$

$$0 \leq \sum_{k=1}^n |z_k - \hat{\eta} e^{i\vartheta} \overline{w_k}|^2 = \sum_{k=1}^n (z_k - \hat{\eta} e^{i\vartheta} \overline{w_k})(\overline{z_k} - \hat{\eta} e^{-i\vartheta} w_k) \quad (1.20)$$

$$= \sum_{k=1}^n |z_k|^2 - 2 \left(\operatorname{Re} e^{-i\vartheta} \sum_{k=1}^n z_k w_k \right) \hat{\eta} + \hat{\eta}^2 \sum_{k=1}^n |w_k|^2 \quad (1.21)$$

$$= a\hat{\eta}^2 - 2b\hat{\eta} + c \quad (1.22)$$

$$\Rightarrow b^2 \leq ac \quad (1.23)$$

Then

$$\left(\operatorname{Re} e^{-i\vartheta} \sum_{k=1}^n z_k w_k \right)^2 \leq \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) \quad (1.24)$$

Suppose $z = \sum_{k=1}^n z_k w_k = |z| e^{i\varphi} \in \mathbb{C}$, let $\vartheta = \varphi$. Then

$$\operatorname{Re} e^{-i\vartheta} \sum_{k=1}^n z_k w_k = \left| \sum_{k=1}^n z_k w_k \right| \quad (1.25)$$

$$\left| \sum_{k=1}^n z_k w_k \right|^2 \leq \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) \quad (1.26)$$

□

1.3 课堂例题 2024 - 02 - 26

1. Let $z_1, z_2 \in \mathbb{C}$, $|z_1| \leq 1$, $|z_2| \leq 1$. If $|z_1 - z_2| \geq 1$, show that

$$|z_1 + z_2| \leq \sqrt{3} \quad (1.27)$$

证明. (平行四边形对角线的平方和等于四边的平方和.)

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = |z_1|^2 + |z_2|^2 - z_1 \overline{z_2} - \overline{z_1} z_2 \quad (1.28)$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + z_1 \overline{z_2} + \overline{z_1} z_2 \quad (1.29)$$

$\Rightarrow$

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) \quad (1.30)$$

$$|z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) - |z_1 - z_2|^2 \leq 3 \quad (1.31)$$

□

2. Let $z_1, \dots, z_n \in \mathbb{C}$, and let $e_0, e_1, \dots, e_{n+1} \in \mathbb{C}$ be the coefficients of $(z+1) \prod_{k=1}^n (z+z_k)$, i.e.

$$(z+1) \prod_{k=1}^n (z+z_k) = \sum_{k=0}^{n+1} e_k z^{n+1-k} \quad (1.32)$$

Show that $\sum_{k=0}^{n+1} (k+1)e_k z^{n+1-k} = 0$ has a root of modulus ≥ 1 .

Specifically, try to show $n=1$ case.

$\Leftrightarrow$ (Let $c \in \mathbb{C}$, show $z^2 + 2(1+c)z + 3c = 0$ has a root of modulus ≥ 1 .)

证明. 下面对方程 $z^2 + 2(1+c)z + 3c = 0$ 的根的情况进行分类 (事实上同时对 $c \in \mathbb{C}$ 的取值进行了分类).

(1) 若方程存在实根 $z_0 \in \mathbb{R}$, 下面可以证明, 事实上 $(1) \Leftrightarrow c \in \mathbb{R}$.

$$z_0^2 + 2(1+c)z_0 + 3c = 0 \quad (1.33)$$

$$\Rightarrow (2z_0 + 3)c = -z_0^2 - 2z_0 \quad (1.34)$$

$$\Rightarrow c = \frac{-z_0^2 - 2z_0}{2z_0 + 3} \in \mathbb{R} \text{ 或 } z_0 = \frac{3}{2} \text{ (此时 } -z_0^2 - 2z_0 \neq 0 \text{ 矛盾)} \quad (1.35)$$

于是 $c \in \mathbb{R}$, $z^2 + 2(1+c)z + 3c = 0$ 为实系数一元二次方程.

$$\Delta = 4(1+c)^2 - 12c = 4(c^2 - c + 1) > 0, \forall c \in \mathbb{R} \quad (1.36)$$

$$z = -1 - c \pm \sqrt{c^2 - c + 1} \in \mathbb{R} \quad (1.37)$$

下面再对实数 $c \in \mathbb{R}$ 的范围分类讨论.

i). $c \geq 0$, 则其中一根 $z = -1 - c - \sqrt{c^2 - c + 1} < -1$, $|z| > 1$.

ii). $c < 0$, 考虑其中一根

$$z = -1 - c - \sqrt{c^2 - c + 1} \quad (1.38)$$

$$= -1 - (\sqrt{c^2 - c + 1} + c) \quad (1.39)$$

由于 $c < 0$, 因此 $1 - c > 0$.

$$\sqrt{c^2 - c + 1} = \sqrt{c^2 + (1 - c)} > \sqrt{c^2} = |c| \quad (1.40)$$

$$\sqrt{c^2 - c + 1} + c > 0 \quad (1.41)$$

$$z = -1 - (\sqrt{c^2 - c + 1} + c) < -1 \quad (1.42)$$

$$|z| > 1 \quad (1.43)$$

于是对于 $\forall c \in \mathbb{R}$, 都有 $|z| > 1$. 从而得证.

事实上, 根据上述证明过程可知, 若 $c \in \mathbb{R}$, 则原方程必有实根, 且两根均为实根, 从而

$$(1): \text{方程存在实根} \Leftrightarrow c \in \mathbb{R} \Leftrightarrow \text{两根均为实根} \quad (1.44)$$

(2) 若方程无实根, 即 $c \in \mathbb{C}$

□

1.4 复数域 $\mathbb{C}$ 上的拓扑概念 & 性质

Let $a \in \mathbb{C}$, open disc of radius r centered at a

$$D_r(a) := \{z \in \mathbb{C} \mid |z - a| < r\} \quad (1.45)$$

$$D_r^*(a) := \{z \in \mathbb{C} \mid 0 < |z - a| < r\} \quad (1.46)$$

closed disc of radius r centered at a

$$\overline{D}_r(a) := \{z \in \mathbb{C} \mid |z - a| \leq r\} \quad (1.47)$$

unit disc :

$$\mathbb{D} := D_1(0) \quad (1.48)$$

Let $\Omega \subseteq \mathbb{C}$

定义 1.4.1. $a \in \Omega$ is an interior point of Ω if $\exists r > 0$, s. t. $D_r(a) \subseteq \Omega$.

注. The set of all interior points of Ω is called the interior of Ω , denoted by $\text{Int}(\Omega)$.

定义 1.4.2. Ω is open if $\Omega = \text{Int}(\Omega)$.

注. $\mathbb{C}$ is open. $\emptyset$ is open. (by convention)

定义 1.4.3. Ω is closed if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

定理 1.4.1. Every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$. That is, $\mathbb{C}$ is Complete.

1.5 课堂例题 2024 – 03 – 01

1.

$$\lim_{n \rightarrow +\infty} z_n = w \Leftrightarrow \lim_{n \rightarrow +\infty} \operatorname{Re} z_n = \operatorname{Re} w, \quad \lim_{n \rightarrow +\infty} \operatorname{Im} z_n = \operatorname{Im} w \quad (1.49)$$

证明.

$$\Rightarrow : |\operatorname{Re} z_n - \operatorname{Re} w| = |\operatorname{Re}(z_n - w)| \leq |z_n - w|$$

$$\Leftarrow : |z_n - w| \leq |\operatorname{Re}(z_n - w)| + |\operatorname{Im}(z_n - w)| = |\operatorname{Re} z_n - \operatorname{Re} w| + |\operatorname{Im} z_n - \operatorname{Im} w|$$

□

2. z is a limit point of $\Omega \Leftrightarrow z$ is an accumulation point of Ω

证明.

$$\Rightarrow : \forall r > 0, \exists N_r, \text{ s. t. } n > N_r, \text{ where } z_n \in \Omega, z_n \neq z.$$

$$z_n \in D_r^*(z), z_n \in \Omega, \forall n > N_r.$$

$$\text{Hence } z_n \in D_r^*(z) \cap \Omega \neq \emptyset, \forall r > 0, n > N_r, \text{ i.e.}$$

z is an accumulation point of Ω

$$\Leftarrow : \text{Take a point } z_n \text{ from } D_{\frac{1}{n}}^*(z) \cap \Omega \text{ which is not empty.}$$

Then $\{z_n\}$ is a Cauchy sequence which converges to z .

Hence z is a limit point of Ω .

□

注. A limit point of Ω may not belong to Ω .

3. 课本第一章练习 T3, T5, T7.

第二章 Week 2 – Functions on $\mathbb{C}$

2.1 连续函数和极值

定义 2.1.1. Let $\Omega \subseteq \mathbb{C}$ be open. We say $f : \Omega \rightarrow \mathbb{C}$ is continuous at $z_0 \in \Omega$ if $\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$\text{whenever } |z - z_0| < \delta, z \in \Omega, \text{ then } |f(z) - f(z_0)| < \epsilon \quad (2.1)$$

To say it another way, $\forall \epsilon > 0, \exists \delta > 0$, s. t. $f(D_\delta(z_0) \cap \Omega) \subseteq D_\epsilon(f(z_0))$

注. We say f is continuous on Ω if f is continuous at every point of Ω .

Here are some facts.

Fact 1. If f is continuous on Ω , then so are $\bar{f}$, $|f|$, $\frac{1}{f}$ (if $f(z) \neq 0$ for all $z \in \Omega$).

证明. For $|f|$, use $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$

□

Fact 2. f is continuous iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous.

命题 2.1.1. Let $\Omega \subseteq \mathbb{C}$ and let f be continuous on Ω . Then

(1) For every open set $S \subseteq \mathbb{C}$, $f^{-1}(S) = \{z \in \Omega \mid f(z) \in S\}$ is open.

(2) For every compact set $K \subseteq \mathbb{C}$, $f(K)$ is compact.

证明.

(1) If $f^{-1}(S) = \emptyset$, true.

Assume $f^{-1}(S) \neq \emptyset$ and let $z_0 \in f^{-1}(S)$. Write $w_0 = f(z_0) \in S$.

Since S is open, $\exists \epsilon > 0$, s. t. $D_\epsilon(w_0) \subseteq S$

Since f is continuous, taking ϵ in the definition, we get a $\delta > 0$, s. t.

$$D_\delta(z_0) \subseteq \Omega \text{ and } f(D_\delta(z_0)) \subseteq D_\epsilon(f(z_0)) = D_\epsilon(w_0) \subseteq S \quad (2.2)$$

Thus $D_\delta(z_0) \subseteq f^{-1}(S)$, and so $f^{-1}(S)$ is open.

(2) Let $\{\Omega_j\}_{j \in J}$ be an open cover of $f(K)$, i.e.

$$f(K) \subseteq \bigcup_{j \in J} \Omega_j \quad (2.3)$$

Then

$$K \subseteq f^{-1}\left(\bigcup_{j \in J} \Omega_j\right) = \bigcup_{j \in J} f^{-1}(\Omega_j) \quad (2.4)$$

By (1), $f^{-1}(\Omega_j)$ is open for all $j \in J$. Thus $\{f^{-1}(\Omega_j)\}_{j \in J}$ is an open cover of K .

Since K is compact, $\exists j_1, \dots, j_n \in J$, s. t.

$$K \subseteq \bigcup_{k=1}^n f^{-1}(\Omega_{j_k}) = f^{-1}\left(\bigcup_{k=1}^n \Omega_{j_k}\right) \quad (2.5)$$

$$\Rightarrow f(K) \subseteq \bigcup_{k=1}^n \Omega_{j_k} \quad (2.6)$$

□

We say that f contains a maximum at $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)|, \quad \forall z \in \Omega \quad (2.7)$$

命题 2.1.2. A continuous function on a compact set Ω is bounded and attains a maximum and a minimum on Ω .

证明. use $|f|^2 = (\operatorname{Re} f)^2 + (\operatorname{Im} f)^2$.

□

2.2 复变函数的极限，全纯函数

定义 2.2.1. Assume $\Omega \subseteq \mathbb{C}$, $\Omega \neq \emptyset$ and $a \in \text{Acc}(\Omega)$, $f : \Omega \rightarrow \mathbb{C}$, $\lim_{z \rightarrow a, z \in \Omega} f(z) = w$ means

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } 0 < |z - z_0| < \delta \Rightarrow |f(z) - w| < \epsilon \quad (2.8)$$

注. 容易证明若极限存在，则极限唯一.

定义 2.2.2. Let $\Omega \subseteq \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$. We say $f(z)$ is Complex differentiable at $z_0 \in \Omega$ if $\lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists. If f is complex differentiable at z_0 , we denote the limit of the quotient by $f'(z_0)$. i.e.

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h} \quad (2.9)$$

$f'(z_0)$ is called the derivative of f at z_0 .

注. If f is complex differentiable at every point of Ω , then we say f is holomorphic on Ω .

例 2.2.1. • $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

• $f(z) = \bar{z}$ is not complex differentiable at any point of $\mathbb{C}$.

• $f(z) = |z|^2$ is only complex differentiable at $z = 0$.

$$\lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0) \Leftrightarrow \lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)-hf'(z_0)}{h} = 0 \quad (2.10)$$

Let $\circ(h)$ denote any complex valued function with the property $\frac{\circ(h)}{h} \rightarrow 0$, as $h \rightarrow 0$

Then f is complex differentiable at z_0 iff $\exists a \in \mathbb{C}$, s.t.

$$f(z_0+h)-f(z_0)-ha = \circ(h), \text{ where } a = f'(z_0) \quad (2.11)$$

注. According to equation(2.11), holomorphic $\Rightarrow$ continuity.

命题 2.2.1. *If f, g are holomorphic on an open set $\Omega \subseteq \mathbb{C}$, then*

$$(f + g)' = f' + g', \quad (fg)' = f'g + fg' \quad (2.12)$$

If $g(z_0) \neq 0$, then $\frac{f}{g}$ is complex differentiable at z_0 and

$$\left(\frac{f}{g}\right)'_{z=z_0} = \frac{f'g - fg'}{g^2} \Big|_{z=z_0} \quad (2.13)$$

If $f : \Omega \rightarrow U$ and $g : U \rightarrow \mathbb{C}$ are holomorphic, then the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z), \quad \forall z \in \Omega \quad (2.14)$$

2.3 Cauchy – Riemann Equations

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \quad (2.15)$$

Assume $\lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists, we may let $h \rightarrow 0$ in whichever manner we please.

(let $z_0 = x_0 + iy_0$)

- Let $h = t \in \mathbb{R}$,

$$f'(z_0) = \lim_{t \rightarrow 0, t \in \mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{h} = u_x(x_0, y_0) + iv_x(x_0, y_0) \quad (2.16)$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \quad (2.17)$$

- Let $h = it, t \in \mathbb{R}$,

$$f'(z_0) = \lim_{t \rightarrow 0, t \in \mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{it} = v_y(x_0, y_0) - iu_y(x_0, y_0) \quad (2.18)$$

$$= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0) \quad (2.19)$$

Thus, we conclude $f = u + iv$ is holomorphic $\Rightarrow u, v$ satisfy

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad (2.20)$$

The equations(2.20) is called Cauchy – Riemann Equations.

例 2.3.1. $f(x + iy) = x^2 - y^2 - 2xyi, x, y \in \mathbb{R}$ is not holomorphic on $\mathbb{C} \setminus \{0\}$.

2.4 全纯条件

Let $f = u + iv : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{on } \Omega \quad (2.21)$$

下面给出函数 *holomorphic* 的充分条件.

定理 2.4.1. Let $\Omega \subset \mathbb{C}$ be open, $f = u + iv : \Omega \longrightarrow \mathbb{C}$. If u, v are differentiable on Ω and satisfy the *Cauchy – Riemann equations*, then f is holomorphic on Ω .

证明. (Goal : $\forall z_0 = x_0 + iy_0 \in \Omega, h = h_1 + ih_2 \in \mathbb{C}, z_0 + h \in \Omega, |h|$ small enough, $f(z_0 + h) - f(z_0) = ah + o(h)$)

Since $u(x, y)$ is differentiable on Ω ,

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = h_1 u_x(x_0, y_0) + h_2 u_y(x_0, y_0) + o(h_1, h_2) \quad (2.22)$$

Here $o(h_1, h_2)$ is any expression with the property that $\frac{o(h_1, h_2)}{\sqrt{h_1^2 + h_2^2}} \rightarrow 0$, as $(h_1, h_2) \rightarrow 0$.

Similarly,

$$v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) = h_1 v_x(x_0, y_0) + h_2 v_y(x_0, y_0) + o(h_1, h_2) \quad (2.23)$$

Then

$$f(z_0 + h) - f(z_0) = h_1 u_x + h_2 u_y + i(h_1 v_x + h_2 v_y) + o(h_1, h_2) \quad (2.24)$$

$$= h_1 u_x - h_2 v_x + i(h_1 v_x + h_2 u_x) + o(h_1, h_2) \quad (2.25)$$

$$= (u_x + iv_x)(h_1 + ih_2) + o(h_1, h_2) \quad (2.26)$$

Note that we may write $o(h)$ instead of $o(h_1, h_2)$, since

$$(h_1, h_2) \rightarrow 0 \Leftrightarrow h \rightarrow 0 \Leftrightarrow |h| \rightarrow 0 \quad (2.27)$$

Then the previous expression is equal to $f'(z_0)h + o(h)$.

Since z_0 is arbitrary, f is holomorphic on Ω . □

$f = u + iv$ can be seen as a mapping

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad (2.28)$$

$$(x, y) \longmapsto (u(x, y), v(x, y)) \quad (2.29)$$

F is said to be differentiable at a point $P_0 = (x_0, y_0)$, if $\exists$ a linear transformation $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, s. t.

$$F(P_0 + H) - F(P_0) = J(H) + |H| \psi(H), \text{ with } |\psi(H)| \rightarrow 0 \text{ as } |H| \rightarrow 0 \quad (2.30)$$

命题 2.4.1. If f is complex differentiable at $z_0 = x_0 + iy_0$, then F is differentiable at (x_0, y_0) .

证明. Since f is complex differentiable at $z_0 = x_0 + iy_0$, we have

$$f(z_0 + h) - f(z_0) = f'(z_0)h + o(h) \quad (2.31)$$

$$= (u_x + iv_x)(h_1 + ih_2) + o(h) \quad (2.32)$$

$$= u_x h_1 - v_x h_2 + i(v_x h_1 + u_x h_2) + o(h) \quad (2.33)$$

$$= u_x h_1 + u_y h_2 + i(v_x h_1 + v_y h_2) + o(h) \quad (2.34)$$

Thus, $F(P_0 + H) - F(P_0) = J(H) + o(H)$, where $J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ and $H = (h_1, h_2)$. □

2.5 复变函数微分

$$z = x + iy, \bar{z} = x - iy \Leftrightarrow x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$$

A given function $f : \Omega \rightarrow \mathbb{C}$ can be expressed either in variables x, y or $z, \bar{z}$. That is, for the given f , we may write $f(x, y)$ or $f(z, \bar{z})$.

注. 可视作复平面上可建立两个坐标系 xOy 和 $zO\bar{z}$, 即 $\mathbb{C}$ 中存在两组基. 由于将复数 z 转化为 $x + iy$ 后再进行计算常常会产生不便, 因此下面通过这两组基之间的转化, 探讨不同形式下函数微分的表达方式.

Suppose the relevant derivatives exist.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f \quad (2.35)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f \quad (2.36)$$

Define two operations. (Wirtinger operations, 1927)

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad (2.37)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad (2.38)$$

命题 2.5.1. *Cauchy – Riemann equations* are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad (2.39)$$

证明. Let $f = u + iv$. Then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} (u_x + v_x + i(u_y + v_y)) = \frac{1}{2} (u_x - v_y + i(u_y + v_x)) \quad (2.40)$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad (2.41)$$

□

注. We note that $f'(z) = u_x + iv_x = u_x - iv_y = 2 \frac{\partial u}{\partial z}$.

调和算子 / 拉普拉斯算子 Define the Laplacian(or the Laplace operator).

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (2.42)$$

注. $C^k(\Omega)$ denotes the set of all k times continuously differentiable functions on Ω .

下面给出调和函数的定义.

定义 2.5.1. Let $\Omega \subset \mathbb{C}$ be an open set. $g : \Omega \longrightarrow \mathbb{C}$ is called harmonic if $g \in C^2(\Omega)$ and $\Delta g = 0$.

下面的命题说明了全纯函数的实部和虚部均调和.(全纯的必要条件)

命题 2.5.2. Let $f = u + iv : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Assume $u, v \in C^2(\Omega)$. Then u, v are harmonic.

注. 事实上后面会证明此处无需 $u, v \in C^2(\Omega)$.

证明. The *Cauchy – Riemann equations* tell
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (2.43)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (2.44)$$

Since $v \in C^2(\Omega)$,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \quad (2.45)$$

Therefore, $u_{xx} + u_{yy} = 0$. Similarly we can proof that $v_{xx} + v_{yy} = 0$. □

A holomorphic function is necessarily harmonic, so is $\bar{f}$.

命题 2.5.3. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \longrightarrow \mathbb{C}$. Then

f is constant iff $f'(z) = 0, \forall z \in \Omega$.

证明.

$\Rightarrow$: clear

$\Leftarrow$: Let $f = u + iv$, then

$$f'(z) = 0 \Rightarrow u_x + iv_x = 0 \Rightarrow u_x = 0, v_x = 0 \quad (2.46)$$

$$\stackrel{C-R}{\Rightarrow} v_y = 0, u_y = 0 \quad (2.47)$$

$$\Rightarrow u = c_1, v = c_2 \text{ (by mean value theorem)} \quad (2.48)$$

$$\text{(区域连通, 利用中值定理)} \quad (2.49)$$

□

2.6 课堂例题 2024 - 03 - 08

1. $f(x + iy) = x^2 - y^2 + 2xyi$ is holomorphic.
2. Is $f(z) = z^2\bar{z} + \frac{1}{z} + \frac{1}{z^2}$ holomorphic on $\mathbb{C} \setminus \{0\}$?
3. Let $f = u + iv$ be holomorphic on a region Ω . Assume $au + bv + c = 0$ for some $a, b, c \in \mathbb{R}$ and a, b are not all zero. Show f is constant.
4. Find a holomorphic function f on $\mathbb{C}$ s. t.

$$\operatorname{Re} f = x^2 - y^2 + xy, f(0) = 0 \quad (2.50)$$

5. Let $\Omega = \mathbb{C} \setminus \{0\}$ and $u : \Omega \rightarrow \mathbb{R}$ be given by $u(x, y) = \frac{1}{2} \ln(x^2 + y^2)$.

Is there a holomorphic function $f : \Omega \rightarrow \mathbb{C}$, s. t. $\operatorname{Re} f = u$?

解. Suppose $f = u + iv$ is holomorphic on Ω . Then

$$\begin{cases} v_x = -u_y = -\frac{y}{x^2+y^2} \\ v_y = u_x = \frac{x}{x^2+y^2} \end{cases} \quad (2.51)$$

By $v_y = \frac{x}{x^2+y^2}$,

$$v = \arctan \frac{y}{x} + c(x) \quad (2.52)$$

Then by $v_x = -\frac{y}{x^2+y^2}$, $c(x) = c$ is constant. $\Rightarrow v = \arctan \frac{y}{x} + c$.

However, $\arctan \frac{y}{x} : \mathbb{R}^2 \rightarrow (-\pi, \pi]$ is not continuous on $\mathbb{R}_{\leq 0} = \{x \leq 0 \mid x \in \mathbb{R}\}$.

(Let $z = x + iy$, then $\arctan \frac{y}{x}$ is an argument of z .)

Therefore, there is no function satisfying the conditions.

注. If the region $\Omega = \mathbb{C} \setminus \{0\}$ is replaced by $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then the answer is yes.

□

6. 课本第一章练习 T8, T9, T10, T13.

第三章 Week 3

3.1 幂级数，解析函数，复对数

与数学分析中的概念一致，下面相当于来复习一下有关幂级数的概念.

- 幂级数 $\sum_{n=0}^{\infty} z_n$ converges $\Leftrightarrow$ 部分和 $\{S_N = \sum_{n=0}^N z_n\}$ converges.
- $\sum_{n=0}^{\infty} |z_n|$ converges $\Rightarrow$ The series converges absolutely(绝对收敛).
- **Absolutely convergent $\Rightarrow$ convergent**
- If $\sum_{n=0}^{\infty} z_n$ converges, then $\lim_{n \rightarrow \infty} z_n = 0$.

A power series (with center 0) is an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ are fixed and z varies in $\mathbb{C}$.(下面通常讨论形式为 $\sum_{n=0}^{\infty} a_n z^n$ 的幂级数)

下面给出复幂级数的收敛半径的定义及收敛圆盘.

定理 3.1.1. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, define

$$R = \lim_{n \rightarrow \infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}} \quad (\text{Hardamard's Formula}) \quad (3.1)$$

(Here we use the convention $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$.) Then

- (1) If $|z| < R$, the series converges absolutely.
- (2) If $|z| > R$, the series diverges.

注. The number R is called the radius of convergence of the power series, and the region $|z| < R$ is called the disc of convergence.

例 3.1.1. 下面给出一些用幂级数定义的常见函数的例子.

- Exponential function

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C} \quad (3.2)$$

- Trigonometric function

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (3.3)$$

- 双曲余弦、正弦

$$\cosh z := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad \sinh z := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (3.4)$$

注. 由定义容易得到, $e^{iz} = \cos z + i \sin z \Rightarrow$ 将 z 限制到 $\mathbb{R}$ 上则有: $e^{i\vartheta} = \cos \vartheta + i \sin \vartheta$.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2} \quad (3.5)$$

下面这个定理说明了幂级数在收敛圆盘内解析. 并给出了幂级数的导数.

定理 3.1.2. The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. Moreover, $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$, which has the same radius of convergence.

证明. Hadamard's formula tells $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same R .

Let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$, $\forall z$ with $|z| < R$, we can find r , s. t. $|z| < r < R$.

For $\forall h \in \mathbb{C}$ s. t. $|h| < r - |z|$, we estimate

$$|f(z+h) - f(z) - hg(z)| = \left| \sum_{n=0}^{\infty} a_n \left((z+h)^n - z^n - nhz^{n-1} \right) \right| \quad (3.6)$$

$$= \left| \sum_{n=2}^{\infty} \left(a_n \sum_{k=2}^n \binom{n}{k} h^k z^{n-k} \right) \right| \quad (3.7)$$

$$\leq |h|^2 \sum_{n=2}^{\infty} |a_n| \sum_{k=0}^{n-2} \binom{n}{k+2} |h^k z^{n-2-k}| \quad (3.8)$$

Since $\binom{n}{k+2} \leq n(n-1)\binom{n-2}{k}$, then

$$|f(z+h) - f(z) - hg(z)| \leq |h|^2 \sum_{n=2}^{\infty} |a_n| n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} |h|^k |z|^{n-2-k} \quad (3.9)$$

$$= |h|^2 \sum_{n=2}^{\infty} |a_n| n(n-1) (|z| + |h|)^{n-2} \quad (3.10)$$

$$< |h|^2 \sum_{n=2}^{\infty} |a_n| n(n-1) r^{n-2} = |h|^2 \cdot c \quad (3.11)$$

Thus

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| < |h| \cdot c \quad (3.12)$$

Therefore, the result follows. $\square$

推论 3.1.3. A power series is infinitely differentiable in its disc of convergence.

注. Thm 3.1.2 即说明了幂级数在收敛圆盘内解析.

下面给出推广到更一般的幂级数的导数, 即中心不一定在原点的情形.

A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \quad (3.13)$$

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f(z) = g(w)$, where $w = z - z_0$.

According to the **Chain Rule**(链式法则), $f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$

下面严格地给出解析的定义.

定义 3.1.1. A function f defined on an open set Ω is said to be analytic at $z_0 \in \mathbb{C}$ if there is a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with positive radius of convergence, s. t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } z \text{ in a neighbourhood of } z_0 \quad (3.14)$$

$$(i.e. \forall z \in D_r(z_0), \text{ for some } r > 0) \quad (3.15)$$

If f is analytic at every point of Ω , then we say f is analytic on Ω .

下面给出有关指数函数 e^z 的一些等式 (命题).

在此之前, 先给出 **Cauchy Multiplication Theorem**.

引理 3.1.4. If $\sum a_n, \sum b_n$ are absolutely convergent, then

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) = \left(\sum a_n \right) \left(\sum b_n \right) \quad (3.16)$$

命题 3.1.1. For $z_1, z_2 \in \mathbb{C}$, $e^{(z_1+z_2)} = e^{z_1} \cdot e^{z_2}$.

推论 3.1.5. If $z = x + iy$, $x, y \in \mathbb{R}$, then

$$e^z = e^x (\cos y + i \sin y) \quad (3.17)$$

推论 3.1.6. De Moire's Formula.

For $\vartheta \in \mathbb{R}$,

$$(\cos \vartheta + i \sin \vartheta)^n = \cos n\vartheta + i \sin n\vartheta \quad (3.18)$$

下面来引入复数域上的对数函数 (**Complex Logarithm**).

$\forall z \in \mathbb{C} \setminus \{0\}$, write $z = re^{i\vartheta}$. Then $e^w = z$ can be solved.

If $w = u + iv$, $u, v \in \mathbb{R}$, then

$$e^u \cdot e^{iv} = re^{i\vartheta} \Rightarrow u = \log r, \quad v = \vartheta + 2k\pi, k \in \mathbb{Z} \quad (3.19)$$

Let $\text{Log}(z)$ be the set of above, then we get Complex Logarithm.

定义 3.1.2. $\forall z \in \mathbb{C} \setminus \{0\}$. Define

$$\text{Log}(z) := \log |z| + i(\arg z + 2k\pi), k \in \mathbb{Z} \quad (3.20)$$

Here $\arg z$ is an argument of z satisfying $-\pi < \arg z \leq \pi$.

(We call $\arg z$ the **principal argument** of z .)

下面介绍复对数的主值支的概念.

定义 3.1.3. Define the principal branch of the logarithm on a "cut plane"

$$\log : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \longrightarrow \mathbb{C} \quad (3.21)$$

$$z \longmapsto \log |z| + i \arg z, \quad -\pi < \arg z < \pi \quad (3.22)$$

例 3.1.2.

$$\operatorname{Log}(-1) = (2k + 1)\pi i \quad (3.23)$$

$$\operatorname{Log}(i) = (2k + \frac{1}{2})\pi i \quad (3.24)$$

$$\log i = \frac{\pi}{2}i \quad (3.25)$$

$$\log(1 + i) = \frac{1}{2} \log 2 + \frac{\pi}{4}i \quad (3.26)$$

命题 3.1.2.

$$e^{\operatorname{Log}(z)} = z, \quad z \neq 0 \quad (3.27)$$

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) \quad (3.28)$$

$$\log z_1 z_2 \neq \log z_1 + \log z_2 \text{ in general} \quad (3.29)$$

3.2 课堂例题 2024 - 03 - 11

1. Let $z \neq 0$. Then $\exists n$ different $z_0, \dots, z_{n-1}$, s. t.

$$z_k^n = z, \quad k = 0, \dots, n-1 \quad (3.30)$$

解. Let $z = |z| e^{i\vartheta}$, $w = r e^{it}$, $r > 0$, $t \in \mathbb{R}$. Then

$$w^n = z \Rightarrow r^n e^{int} = |z| e^{i\vartheta} \Rightarrow \begin{cases} r = |z|^{\frac{1}{n}} \\ nt = \vartheta + 2k\pi, k \in \mathbb{Z} \end{cases} \quad (3.31)$$

□

2. Proof

$$\left| \sum_{k=0}^n e^{ikx} \right| \leq \left| \frac{1}{\sin \frac{x}{2}} \right|, \quad \forall x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\} \quad (3.32)$$

3. 课本第一章练习 T16, T19

3.3 复对数的性质

Let $a \in \mathbb{C}$. We may define

$$z^a = e^{a \log z}, \quad z \neq 0 \quad (3.33)$$

命题 3.3.1. The function $f(z) = \log z$, $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is holomorphic.

证明. $\forall z_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, let $w = \log z$, $w_0 = \log z_0$. Then

$$\lim_{z \rightarrow z_0} \frac{\log z - \log z_0}{z - z_0} = \lim_{w \rightarrow w_0} \frac{w - w_0}{e^w - e^{w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0} \quad (3.34)$$

Therefore, $(\log z)' = \frac{1}{z}$. □

命题 3.3.2. Show

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \text{ on } \mathcal{D} \quad (3.35)$$

证明. Let $f(z) = \log(1+z)$, $g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$. Both are holomorphic on $\mathcal{D}$ and

$$f'(z) = \frac{1}{1+z}, \quad g'(z) = \sum_{n=1}^{\infty} (-1)^n z^{n-1} = \frac{1}{1+z} \quad (3.36)$$

And so $(f - g)' = 0$ on $\mathcal{D}$. Therefore, $f - g = c$. Taking $z = 0$, $f(0) = g(0) \Rightarrow c = 0$. □

3.4 道路

先给出道路 (path) 的定义.

定义 3.4.1. A continuous function $z(t) = x(t) + iy(t)$ from $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$ is called a path (or a parametric curve) connecting $z(a)$ and $z(b)$. ($z(a)$ is called the starting point, $z(b)$ the end point) The path is closed if $z(a) = z(b)$.

The path is simple if $z(t) \neq z(s)$ unless $\begin{cases} (1) t = s \\ (2) t = a, s = b \end{cases}$

下面给出道路光滑性的描述.

定义 3.4.2. We say that a path $z(t) = x(t) + iy(t)$, $t \in [a, b]$ is smooth if $x(t)$, $y(t)$ are continuously differentiable and $z'(t) = x'(t) + iy'(t) \neq 0$, $t \in [a, b]$. Here $z'(a)$, $z'(b)$ are understood as one-sided derivative.

下面给出两条道路等价的定义.

定义 3.4.3. Two paths $z : [a, b] \rightarrow \mathbb{C}$, $\tilde{z} : [c, d] \rightarrow \mathbb{C}$ are equivalent if $\exists$ bijection and differential

$$t : [c, d] \rightarrow [a, b] \quad (3.37)$$

$$s \mapsto t(s) \quad (3.38)$$

s. t. $\tilde{z}(s) = z(t(s))$ and $t'(s) > 0$.

下面给出道路反向的定义.

定义 3.4.4. Given a path z , we can define a path $\tilde{z}$ obtained from z by reversing the orientation

$$z(t) : [a, b] \rightarrow \mathbb{C} \quad (3.39)$$

$$\tilde{z}(t) = z(a + b - t) : [a, b] \rightarrow \mathbb{C} \quad (3.40)$$

这里我们规定一下道路的正向 / 逆向 (逆时针为正向).

定义 3.4.5. A path has positive orientation if it travels counterclockwisely.

($\cdots$ negative orientation $\cdots$ clockwise.)

下面我们给出分段光滑的定义.

定义 3.4.6. A path $z(t) : [a, b] \rightarrow \mathbb{C}$ is piecewise smooth if $\exists$ a partion $a = a_0 < a_1 < \cdots < a_n = b$, s. t. $z(t)$ is smooth in each $[a_k, a_{k+1}]$, $k = 0, \cdots, n-1$.

下面说明两条道路的连接.

Paths can be concatenated. If $z : [a, b] \rightarrow \mathbb{C}$, $\tilde{z} : [b, c] \rightarrow \mathbb{C}$ and $z(b) = \tilde{z}(b)$, we can define $w : [a, c] \rightarrow \mathbb{C}$ as $w(t) = \begin{cases} z(t), & a \leq t \leq b \\ \tilde{z}(t), & b \leq t \leq c \end{cases}$. Concatenation of $z, \tilde{z}$ is denoted as $z \circ \tilde{z}$.

下面给出 **zig-zag** 道路的定义.

定义 3.4.7. A path is zig-zag if it consists of finitely many horizontal or vertical line segments.

下面的命题说明区域内的任两点可由一条 zig-zag 道路连接.

命题 3.4.1. Let $\Omega \subset \mathbb{C}$ be a region. Then any two points in Ω can be joined by a zig-zag path.

证明.

- Case when $\Omega = D_R(z_0)$, where $z_0 \in \mathbb{C}$, $R > 0$.
 $\forall a, \beta \in \Omega$, we can join them to the horizontal diameter via a vertical line segment.
- Now let Ω be an arbitrary region. $\forall a \in \Omega$. Let

$$A := \{\beta \in \Omega \mid \exists \text{ zig-zag path } \gamma \text{ connecting } \beta \text{ and } a\} \quad (3.41)$$

Then 容易证 $a \in A \neq \emptyset$ 既开又闭, 从而 $A = \Omega$.

□

3.5 课堂例题 2024 – 03 – 15

1. Calculate $2^i, i^i$.
2. Find all possible values of $(1 + \sqrt{3}i)^{\frac{1}{8}}$.
3. Let $z_n \in \mathbb{C}$, $\operatorname{Re} z_n \geq 0$, $n = 1, 2, \dots$. If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_n^2$ both converge, show that $\sum_{n=1}^{\infty} |z_n|^2$ converges.
4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be holomorphic on $\mathcal{D}$. Assume $|f(z)| \leq 1$, $\forall z \in \mathcal{D}$. Show $|a_n| \leq 1$, $n = 1, 2, \dots$.

第四章 Week 4

4.1 曲线积分

积分 下面先给出复数域上积分的定义.

定义 4.1.1. Let $z(t) = x(t) + iy(t)$, $t \in [a, b] \subset \mathbb{R}$. If $x(t), y(t)$ are differentiable, we define $z'(t) = x'(t) + iy'(t)$.

Similarly, if $x(t), y(t)$ are continuous, we define

$$\int_a^b z(t)dt = \int_a^b x(t)dt + i \int_a^b y(t)dt \quad (4.1)$$

容易证明, 复数域上的积分同样具有三角不等式.

命题 4.1.1. Let $f : [a, b] \rightarrow \mathbb{C}$ be continuous. Then

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)| dt \quad (4.2)$$

证明. Write $\int_a^b f(t)dt = re^{i\theta}$, $r \geq 0$. Then

$$r = e^{-i\theta} \int_a^b f(t)dt = \int_a^b e^{-i\theta} f(t)dt = \left| \int_a^b \operatorname{Re} e^{-i\theta} f(t)dt \right| \quad (4.3)$$

$$\leq \int_a^b |\operatorname{Re} e^{-i\theta} f(t)| dt \quad (4.4)$$

$$\leq \int_a^b |e^{-i\theta} f(t)| dt = \int_a^b |f(t)| dt \quad (4.5)$$

□

曲线积分 下面给出复数域上连续道路的曲线积分的定义.

定义 4.1.2. Let $\Omega \subset \mathbb{C}$ be open. Given a smooth path γ in Ω parametrized by $z : [a, b] \rightarrow \Omega$ and a continuous function $f : \Omega \rightarrow \mathbb{C}$. We define the integral of f along γ by

$$\int_{\gamma} f(z) dz := \int_a^b f(z(t)) z'(t) dt \quad (4.6)$$

Let $\tilde{z}(t) : [c, d] \rightarrow \Omega$ be equivalent to $z(t)$. Then

$$\int_a^b f(z(t)) z'(t) dt = \int_c^d f(\tilde{z}(t)) \tilde{z}'(t) dt \quad (4.7)$$

下面给出分段连续道路的曲线积分及曲线长度的定义.

定义 4.1.3. If γ is piecewise smooth and $z(t)$ is a piecewise smooth parametrization as before, we define

$$\int_{\gamma} f(z) dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t)) z'(t) dt \quad (4.8)$$

The length of the smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |z'(t)| dt \quad (4.9)$$

If $f = u + iv$, $z(t) = x(t) + iy(t)$, then

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt = \int_a^b (u + iv)(x'(t) + iy'(t)) dt \quad (4.10)$$

$$= \int_a^b (ux'(t) - vy'(t)) dt + i \int_a^b (vx'(t) + uy'(t)) dt \quad (4.11)$$

$$= \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (v dx + u dy) \quad (4.12)$$

下面给出曲线积分的几条性质.

命题 4.1.2. 记 γ^- 为 γ 的反向.

$$(1) \int_{\gamma} f(z) dz = - \int_{\gamma^-} f(z) dz.$$

(2) If $f(z), g(z)$ are continuous, and γ is a path, then $\forall a, \beta \in \mathbb{C}$,

$$\int_{\gamma} (af + \beta g) dz = a \int_{\gamma} f dz + \beta \int_{\gamma} g dz \quad (4.13)$$

(3)

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{\gamma} |f(z)| \cdot \text{length}(\gamma) \quad (4.14)$$

原函数 下面我们给出原函数的概念.

定义 4.1.4. If $f : \Omega \rightarrow \mathbb{C}$. Assume $\exists$ a complex differentiable $F : \Omega \rightarrow \mathbb{C}$, s. t.

$$F'(z) = f(z), \text{ for every } z \in \Omega \quad (4.15)$$

Then we say f admits a primitive (or an antiderivative) on Ω .

下面的命题说明若函数有原函数, 则其曲线积分只与始末点有关, 而与路径无关.

命题 4.1.3. If f is a continuous function that admits a primitive F on Ω , and γ is a path in Ω that begins at w_1 and ends at w_2 , then

$$\int_{\gamma} f(z) dz = F(w_2) - F(w_1) \quad (4.16)$$

证明. Let $z(t) : [a, b] \rightarrow \Omega$ be a parametrization for γ with $z(a) = w_1$, $z(b) = w_2$.

- Assume γ is smooth. Compute

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt = \int_a^b F'(z(t)) z'(t) dt = \int_a^b \frac{dF(z(t))}{dt} dt \quad (4.17)$$

According to **the fundamental theorem of calculus**, we get

(分别对实部和虚部运用微积分基本定理)

$$\int_{\gamma} f(z) dz = \int_a^b F'(z(t)) z'(t) dt = \int_a^b \frac{dF(z(t))}{dt} dt \quad (4.18)$$

$$= F(z(b)) - F(z(a)) = F(w_2) - F(w_1) \quad (4.19)$$

- γ is piecewise smooth, we can proof similarly.

□

由命题 4.1.3, 可得到有原函数的函数 f 在闭曲线上积分为 0.

推论 4.1.1. If γ is a closed path in Ω , f is continuous and admits a primitive on Ω , then

$$\int_{\gamma} f(z) dz = 0 \quad (4.20)$$

同时, 由命题 4.1.3, 还可得到区域 Ω 上导数恒为 0 的全纯函数只能为常值函数.

推论 4.1.2. If f is holomorphic on a region Ω and $f' \equiv 0$, then f is constant.

下面给出具有原函数的充要条件.

定理 4.1.3. Let $\Omega \subset \mathbb{C}$ be a region. $f : \Omega \rightarrow \mathbb{C}$ be a continuous function. Then the following statements are equivalent:

- (1) f admits a primitive on Ω .
- (2) $\forall a, \beta \in \mathbb{C}$, $\int_{\gamma} f(z) dz$ is invariant for any path γ in Ω that joins a to β .
- (3) $\forall a, \beta \in \mathbb{C}$, $\int_{\gamma} f(z) dz$ is invariant for any zig-zag path γ in Ω that joins a to β .

注. 我们将在定理 5.2.5 中补充一条具有原函数的充要条件. (详见 **Thm 5.2.5 (4)**)

证明. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1): Fix $a \in \Omega$ and define $F : \Omega \rightarrow \mathbb{C}$ by

$$F(z_0) = \int_{\gamma} f(z) dz, \quad z_0 = x_0 + iy_0 \in \Omega \quad (4.21)$$

where γ is any zig-zag path joining a to z_0 .

(**F is Well-defined** : Condition (3) tells $F(z_0)$ is independent of the choice of γ .)

Let $F(z) = U + iV$, $f(z) = u + iv$. It suffices to show

$$\begin{cases} U_x(x_0, y_0) = u(x_0, y_0), & V_x(x_0, y_0) = v(x_0, y_0) \\ U_y(x_0, y_0) = -v(x_0, y_0), & V_y(x_0, y_0) = u(x_0, y_0) \end{cases} \quad (4.22)$$

- $U_x(x_0, y_0) = u(x_0, y_0), \quad V_x(x_0, y_0) = v(x_0, y_0)$

Let $h \in \mathbb{R}$. Let γ be a zig-zag path joining a to z_0 ,

$\gamma_H : z_H(t) = z_0 + th, 0 \leq t \leq 1, \gamma_H \subset \Omega$. Then

$$F(z_0 + h) = \int_{\gamma \circ \gamma_H} f(z) dz = \int_{\gamma} f(z) dz + \int_{\gamma_H} f(z) dz \quad (4.23)$$

$$= F(z_0) + \int_{\gamma_H} f(z) dz \quad (4.24)$$

Then we get

$$\frac{F(z_0 + h) - F(z_0)}{h} = \int_0^1 f(z_0 + th) dt \quad (4.25)$$

Since f is continuous,

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{F(z_0 + h) - F(z_0)}{h} = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \int_0^1 f(z_0 + th) dt \quad (4.26)$$

$$= f(z_0) \quad (4.27)$$

$$= u(x_0, y_0) + iv(x_0, y_0) \quad (4.28)$$

- $U_y(x_0, y_0) = -v(x_0, y_0)$, $V_y(x_0, y_0) = u(x_0, y_0)$

Similarly.

□

4.2 课堂例题 2024 - 03 - 18

1. Let $f(x + iy) = x$. Consider two paths γ_1, γ_2 joining 0 to 1.

$$z_1(t) = t, \quad 0 \leq t \leq 1 \quad (4.29)$$

$$z_2(t) = \begin{cases} t + 2ti, & 0 \leq t \leq \frac{1}{2} \\ t + 2(1-t)i, & \frac{1}{2} \leq t \leq 1 \end{cases} \quad (4.30)$$

Evaluate

$$\int_{\gamma_1} f(z)dz, \quad \int_{\gamma_2} f(z)dz \quad (4.31)$$

$$\left(= \frac{1}{2}, \quad = \frac{1-i}{2} \right) \quad (4.32)$$

2. Let $f(z) = \frac{1}{z}$, $z \in \mathbb{C} \setminus \{0\}$. Let $\gamma : z(t) = Re^{it}$, $R > 0$, $0 \leq t \leq 2\pi$.

Evaluate

$$\int_{\gamma} f(z)dz \quad (= 2\pi i) \quad (4.33)$$

3. Let $f(z) = z^3$ and let σ be any path joining 1 to $2 + i$. Evaluate

$$\int_{\gamma} f(z)dz \quad (4.34)$$

4. 课本第一章练习 T26.

4.3 Cauchy's Theorem

这节课我们来介绍一个重要的定理——**Cauchy's Theorem**.

单连通 下面先给出一个定理并借此给出曲线的**内部**和**外部**的定义.

定理 4.3.1. Jordan Curve Theorem.

Let γ be a simple closed curve on $\mathbb{C}$. Then $\mathbb{C} \setminus \gamma$ has two connected components. The bounded component is called the interior of γ and the unbounded component is called the exterior of γ .

If the simple closed path γ is positively oriented, then $\text{Interior}(\gamma)$ is to the **left** while traversing γ .

证明. 证明见书¹Page 351

□

下面给出**单连通集**的定义.

定义 4.3.1. A region $\Omega \subset \mathbb{C}$ is simply connected if for every closed path $\gamma \subset \Omega$, $\text{Interior}(\gamma) \subset \Omega$.

例 4.3.1. 下面给出几个常见的单连通集 / 非单连通集的例子.

- $\mathbb{C}$, $D_r(z_0)$, $r > 0$, $z_0 \in \mathbb{C}$ are simply connected.
- $\mathbb{C} \setminus \{0\}$, $D_r^*(z_0)$ are not simply connected.
- $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is simply connected.

¹课堂教材: 《Complex Analysis》— Elias M. Stein

Cauchy's Theorem 下面介绍 **Cauchy's Theorem**.

定理 4.3.2. Let $\Omega \subset \mathbb{C}$ be **simply connected**, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Then for any closed path γ ,

$$\int_{\gamma} f(z) dz = 0 \quad (4.35)$$

注. 条件中 Ω 单连通 **不可省略**. 下面给出反例.

例 4.3.2. Let $\Omega = \mathbb{C} \setminus \{0\}$, $f(z) = \frac{1}{z}$. Then f is holomorphic on Ω and $(C_1(0))$ 表示单位圆周)

$$\int_{C_1(0)} f(z) dz = 2\pi i \neq 0 \quad (4.36)$$

证明. The result would follow if f has a primitive on Ω .

By Thm 4.1.3, thus it suffices to show that for any two zig-zag paths γ_1, γ_2 having the same starting points and ending points,

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz \quad (4.37)$$

$$\text{i.e. } \int_{\gamma_1 \circ \gamma_2^{-1}} f(z) dz = 0 \quad (4.38)$$

Equivalently, we have to show $\int_{\gamma} f(z) dz = 0$ for any closed zig-zag path γ .

By concatenate some horizontal or vertical paths, any closed zig-zag path is the union of rectangle paths.

Thus we are done if we can show $\int_R f(z) dz = 0$, where R is a rectangle path.

Note that

$$\int_R f(z) dz = \int_{T_1} f(z) dz + \int_{T_2} f(z) dz \quad (4.39)$$

Then the theorem boils down to showing $\int_T f(z) dz = 0$ for any triangle path T in Ω .

(This is Coursat Theorem on P34.)

□

下面给出 **Cauchy's Theorem** 的另一种叙述, 这里并不对 f 的定义域 Ω 做单连通要求.

定理 4.3.3. Let γ be a simple closed path. If f is holomorphic in $\text{Interior}(\gamma)$ and continuous on γ , then

$$\int_{\gamma} f(z) dz = 0 \quad (4.40)$$

4.4 课堂例题 2024 – 03 – 22

1. 课本第一章练习 T_{25} .
2. 课本第二章练习 T_5, T_6 .

第五章 Week 5

5.1 闭路变形原理

围道 为了叙述方便, 我们将简单闭曲线记作**围道 (contour)**. 并默认其为正向.

定义 5.1.1. A simple closed path is called a **contour**. If nothing is specified, we'll assume the contour is positively oriented.

闭路变形原理 下面来介绍闭路变形原理.(实际上可视为 Cauchy's Theorem 的推论)

定理 5.1.1. Principle of contour deformation.

Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Let γ_1 be a contour in Ω , γ_2 be another contour in $\Omega \cap \text{Interior}(\gamma_1)$. If $\text{Interior}(\gamma_1) \cap \text{Exterior}(\gamma_2) \subset \Omega$, then

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz \quad (5.1)$$

注. 条件 $\text{Interior}(\gamma_1) \cap \text{Exterior}(\gamma_2) \subset \Omega$ 保证了 γ_1 与 γ_2 之间围成的区域不存在空洞 (亏格), 从而在这片区域上 Cauchy's Theorem 总是奏效的.



图 5.1: Principle of contour deformation

证明.

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2^-} f(z)dz \quad (5.2)$$

$$= \int_{\mu_1} f(z)dz + \int_{\mu_2} f(z)dz = 0 + 0 = 0 \quad (5.3)$$

□

下面给出 Thm 5.1.1 的一种 Special Case. ($\text{Interior}(\gamma_1)$ 仅有一点不在 Ω 内)

推论 5.1.2. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Let γ be a contour in Ω and $\exists a \in \text{Interior}(\gamma)$, s. t. $\text{Interior}(\gamma) \setminus \{a\} \subset \Omega$. Then

$$\int_{\gamma} f(z)dz = \int_{C_r(a)} f(z)dz, \text{ where } C_r(a) \subset \text{Interior}(\gamma) \quad (5.4)$$

下面给出更一般的表述. ($\text{Interior}(\gamma)$ 内有有限个点不在 Ω 内)

推论 5.1.3. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic and γ is a contour in Ω . If $a_1, \dots, a_n \in \text{Interior}(\gamma)$, such $\text{Interior}(\gamma) \setminus \{a_1, \dots, a_n\} \subset \Omega$, then

$$\int_{\gamma} f(z)dz = \sum_{k=1}^n \int_{C_{r_k}(a_k)} f(z)dz \quad (5.5)$$

where $C_{r_k}(a_k)$ are disjoint “small” circles.

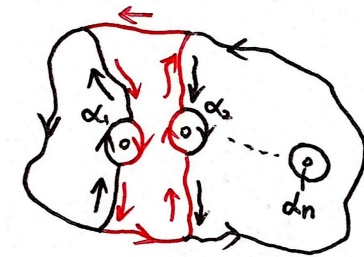


图 5.2: Principle of contour deformation (Special Case)

证明. 即证

$$\int_{\gamma} f(z)dz + \sum_{k=1}^n \int_{C_{r_k}^-(a_k)} f(z)dz = \sum_{k=1}^n \int_{\mu_k} f(z)dz = \sum_{k=1}^n 0 = 0 \quad (5.6)$$

□

5.2 Cauchy Integral Formulas

接下来我们介绍另一个计算环路积分非常重要的公式——**Cauchy Integral Formulas**.

Cauchy Integral Formulas 首先给出一个命题.

命题 5.2.1. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic, γ be a contour in Ω . Assume $\exists a \in \text{Interior}(\gamma)$, s. t.

$$\text{Interior}(\gamma) \setminus \{a\} \subset \Omega \text{ and } \lim_{z \rightarrow a} (z - a)f(z) = 0 \quad (5.7)$$

Then

$$\int_{\gamma} f(z) dz = 0 \quad (5.8)$$

证明. $\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$|z - a| < \delta \Rightarrow |(z - a)f(z)| < \epsilon \quad (5.9)$$

By the **principle of contour deformation** (Thm 5.1.1),

$$\int_{\gamma} f(z) dz = \int_{C_r(a)} f(z) dz, \text{ where } 0 < r < \delta \quad (5.10)$$

Then

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in C_r(a)} |f(z)| \cdot \text{length}(\gamma) = \sup_{z \in C_r(a)} |f(z)| \cdot 2\pi r \quad (5.11)$$

$$< \frac{\epsilon}{r} \cdot 2\pi r = 2\pi\epsilon \quad (5.12)$$

□

下面给出 **Cauchy Integral Formulas**.

定理 5.2.1. CIF.

Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Then for any $z_0 \in \Omega$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta \quad (5.13)$$

where γ is any contour in Ω s. t. $z_0 \in \text{Interior}(\gamma) \subset \Omega$.

证明. Let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \quad (5.14)$$

Then $g(z)$ is holomorphic on $\Omega \setminus \{z_0\}$. Clearly

$$\lim_{z \rightarrow z_0} (z - z_0)g(z) = 0, \quad \text{Interior}(\gamma) \setminus \{z_0\} \subset \Omega \setminus \{z_0\} \quad (5.15)$$

By Prop 5.2.1,

$$\int_{\gamma} g(z) dz = 0 \Rightarrow \int_{\gamma} \frac{f(z)}{z - z_0} dz = \int_{\gamma} \frac{f(z_0)}{z - z_0} dz \quad (5.16)$$

Therefore

$$\int_{\gamma} \frac{f(z)}{z - z_0} dz = \int_{\gamma} \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_{\gamma} \frac{1}{z - z_0} dz = f(z_0) \int_{C_r(z_0)} \frac{1}{z - z_0} dz \quad (5.17)$$

$$= 2\pi i \cdot f(z_0) \quad (5.18)$$

□

Therefore, according to Cauchy Integral Formulas, we get

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (5.19)$$

where f is holomorphic on an open set containing the contour γ and its interior, $z \in \text{Interior}(\gamma)$.

高阶 Cauchy Integral Formulas 下面给出高阶的 Cauchy Integral Formulas.

定理 5.2.2. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Then f is complex differentiable to all orders and moreover, $\forall z \in \Omega$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n = 0, 1, 2, \dots \quad (5.20)$$

where γ is any contour in Ω s. t. $z \in \text{Interior}(\gamma) \subset \Omega$.

Cauchy's Inequalities 作为柯西积分公式 (CIF) 的推论, 下面可得到 **Cauchy's Inequalities**.

推论 5.2.3. Cauchy's Inequalities.

If f is holomorphic on an open set Ω with $D_R(z_0) \subset \Omega$, $R > 0$, then

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_{C_R(z_0)}}{R^n}, \quad \text{where } \|f\|_{C_R(z_0)} = \sup_{z \in C_R(z_0)} |f(z)| \quad (5.21)$$

Liouville's theorem 下面给出 **CIF** 的另一条重要推论. 在此之前, 先给出**整函数 (entire)** 的定义.

定义 5.2.1. A holomorphic function defined on the whole $\mathbb{C}$ is called an **entire function**.

下面给出刘维尔定理 (**Liouville's theorem**) 的内容.

推论 5.2.4. Liouville's theorem.

If f is entire and bounded, then f is constant.

证明. $\forall z \in \mathbb{C}$, by **Cauchy's Inequalities (Cor 5.2.3)**

$$|f'(z)| \leq \frac{\|f\|_{\mathbb{C}}}{R} \leq \frac{M}{R}, \quad \text{where } |f| \leq M \quad (5.22)$$

Letting $R \rightarrow \infty$, we get $f'(z) = 0, \forall z \in \mathbb{C} \Rightarrow f$ is constant. □

具有原函数的充要条件 下面在定理 4.1.3 的基础上，再增加一条具有原函数的充要条件.

定理 5.2.5. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \rightarrow \mathbb{C}$ be continuous. Then **TFAE** (the followings are equivalent):

- (1) f admits a primitive on Ω .
- (2) $\forall a, \beta \in \Omega$, $\int_{\gamma} f(z)dz$ is invariant for any path.
- (3) $\forall a, \beta \in \Omega$, $\int_{\gamma} f(z)dz$ is invariant for any zig-zag path.
- (4) $\int_{\gamma} f(z)dz = 0$ for any closed path $\gamma \subset \Omega$.

证明. (4) $\Rightarrow$ (2): Fix $a, \beta \in \Omega$, $\forall$ two paths $\gamma_1, \gamma_2 \subset \Omega$ joining a to β , by (4)

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1 \circ \gamma_2^-} f(z)dz = 0, \text{ where } \gamma_1 \circ \gamma_2^- \text{ is a closed path in } \Omega \quad (5.23)$$

Therefore

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz, \quad \forall \gamma_1, \gamma_2 \subset \Omega \quad (5.24)$$

□

Morera's Theorem 有了具有原函数的充要条件之后 (Thm 5.2.5)，下面我们可以对全纯函数进行充要刻画，此即为 **Morera's Theorem** 的推广.

命题 5.2.2. Let $\Omega \subset \mathbb{C}$ be **simply connected**, $f : \Omega \rightarrow \mathbb{C}$ be continuous. Then **TFAE**

- (1) f is holomorphic on Ω .
- (2) $\int_{\gamma} f(z)dz = 0$ for any closed path $\gamma \subset \Omega$.

注. 命题中 (2) $\Rightarrow$ (1) 的部分即为 **Morera's Theorem**.

证明. (1) $\Rightarrow$ (2): is by **Cauchy's Theorem** (Thm 4.3.2).

(2) $\Rightarrow$ (1): By Thm 5.2.5, f admits a primitive F on Ω , i.e. $F' = f$, then F is holomorphic on Ω .

By **CIF** (Thm 5.2.2), F is infinitely complex differentiable.

In particular, F is twice complex differentiable. $\Rightarrow F'' = f' \Rightarrow f$ is holomorphic on Ω .

□

由命题 5.2.2 可直接得到如下推论.

推论 5.2.6. Every holomorphic function on a simply connected region admits a primitive.

在利用命题 5.2.2 (Morera's Theorem) 判定函数全纯时, 要注意条件 (2) 中道路 γ 的任意性. 下面便给出一个反例.

例 5.2.1. Suppose $\int_{C_r(0)} f(z) dz = 0$ for all $0 < r < 1$, can we conclude f is holomorphic on $\mathbb{D}$?

解. The answer is absolutely No. Take $f(z) = |z|^2$. Then $\int_{C_r(0)} f(z) dz = 0, \forall 0 < r < 1$.

Since

$$f(z) = |z|^2 = z \cdot \bar{z}, \quad \frac{\partial f}{\partial \bar{z}} = z \neq 0, \quad \forall z \in \mathbb{C} \setminus \{0\} \quad (5.25)$$

Therefore, f is not holomorphic on $\mathbb{D}$.

注. 事实上, 取道路 γ 为以原点为圆心, $\frac{1}{2}$ 为半径的上半圆, 逆时针方向, 可证 $\int_{\gamma} f(z) dz \neq 0$.

□

5.3 课堂例题 2024 – 03 – 25

1. Evaluate

$$\int_{\gamma} \frac{2z-1}{z^2-z} dz \quad (5.26)$$

where γ is any contour s. t. $\overline{D} \subset \text{Interior}(\gamma)$.

解. By Cor 5.1.3, (闭路变形原理)

$$\int_{\gamma} f(z) dz = \int_{C_{\frac{1}{3}}(0)} f(z) dz + \int_{C_{\frac{1}{3}}(1)} f(z) dz \quad (5.27)$$

$$= \int_{C_{\frac{1}{3}}(0)} \left(\frac{1}{z-1} + \frac{1}{z} \right) dz + \int_{C_{\frac{1}{3}}(1)} \left(\frac{1}{z-1} + \frac{1}{z} \right) dz \quad (5.28)$$

Since $\frac{1}{z-1}$ is holomorphic on $\text{Interior}(C_{\frac{1}{3}}(0)) = D_{\frac{1}{3}}(0)$, where $D_{\frac{1}{3}}(0)$ is simply connected, then by Cauchy's Theorem,

$$\int_{C_{\frac{1}{3}}(0)} \left(\frac{1}{z-1} + \frac{1}{z} \right) dz = \int_{C_{\frac{1}{3}}(0)} \frac{1}{z} dz \quad (5.29)$$

Similarly, since $\frac{1}{z}$ is holomorphic on $D_{\frac{1}{3}}(1)$, then

$$\int_{C_{\frac{1}{3}}(1)} \left(\frac{1}{z-1} + \frac{1}{z} \right) dz = \int_{C_{\frac{1}{3}}(1)} \frac{1}{z-1} dz \quad (5.30)$$

Therefore

$$\int_{\gamma} f(z) dz = \int_{C_{\frac{1}{3}}(0)} \left(\frac{1}{z-1} + \frac{1}{z} \right) dz + \int_{C_{\frac{1}{3}}(1)} \left(\frac{1}{z-1} + \frac{1}{z} \right) dz \quad (5.31)$$

$$= \int_{C_{\frac{1}{3}}(0)} \frac{1}{z} dz + \int_{C_{\frac{1}{3}}(1)} \frac{1}{z-1} dz \quad (5.32)$$

$$= 2\pi i + 2\pi i \quad (5.33)$$

$$= 4\pi i \quad (5.34)$$

□

2. Evaluate

$$\int_{C_3(0)} \frac{e^{z^2}}{z-2} dz \quad (= 2\pi i f(z_0) = 2\pi i e^4) \quad (5.35)$$

$$\int_{C_1(0)} \frac{e^{z^2}}{z-2} dz \quad (= 0) \quad (5.36)$$

3. Evaluate

$$\int_{C_2(0)} \frac{\sin z}{z^2 + 1} dz \quad (5.37)$$

解.

$$\int_{C_2(0)} \frac{\sin z}{z^2 + 1} dz = \int_{C_{\frac{1}{2}}(i)} \frac{\sin z}{z+i} \cdot \frac{1}{z-i} dz + \int_{C_{\frac{1}{2}}(-i)} \frac{\sin z}{z-i} \cdot \frac{1}{z+i} dz \quad (5.38)$$

$$= 2\pi i \cdot \frac{\sin i}{2i} + 2\pi i \cdot \frac{\sin(-i)}{-2i} \quad (5.39)$$

$$= 2\pi \sin i \quad (5.40)$$

□

4. Evaluate

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz \quad (5.41)$$

解. By Thm 5.2.2 (高阶 Cauchy Integral Formulas),

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz = \int_{C_{\frac{1}{3}}(1)} \frac{z}{(z-2)^2} \cdot \frac{1}{z-1} dz + \int_{C_{\frac{1}{3}}(2)} \frac{z}{z-1} \cdot \frac{1}{(z-2)^2} dz \quad (5.42)$$

$$= 2\pi i + 2\pi i \left(\frac{z}{z-1} \right)' \Big|_{z=2} \quad (5.43)$$

$$= 0 \quad (5.44)$$

□

5. 课本第二章练习 T6.

5.4 Sequence of functions

概念 这一节我们来介绍有关复数域上函数列的相关概念和性质. 先回顾函数列收敛的定义.

定义 5.4.1. Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \rightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of function.

We say $\{f_n\}_{n=1}^{\infty}$ **converges** if $\forall z \in \Omega$, $\{f_n(z)\}_{n=1}^{\infty}$ converges.

We say $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$m, n > N \Rightarrow |f_m(z) - f_n(z)| < \epsilon, \quad \forall z \in \Omega$$

We say $\{f_n\}_{n=1}^{\infty}$ **uniformly converges to f** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$n > N \Rightarrow |f_n(z) - f(z)| < \epsilon, \quad \forall z \in \Omega$$

下面给出一个经典的收敛但不一致收敛的例子.

例 5.4.1. $\{f_n(z) = z^n\}_{n=1}^{\infty}$ on $\mathbb{D}$ is convergent but not uniformly convergent. However, the sequence is uniformly convergent on any compact subset on $\mathbb{D}$.

一致收敛 下面给出函数列一致收敛的性质, 即积分与极限可交换次序.

定理 5.4.1. Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \rightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of continuous functions that converges uniformly to f **on every compact subset of Ω** . Then

(1) f is continuous.

(2) If $\gamma \subset \Omega$ is a path with finite length, then

$$\lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} f(z) dz \quad (5.45)$$

(3) If f_n is holomorphic for all n , then so is f . Moreover,

$\{f'_n\}_{n=1}^{\infty}$ converges uniformly to f' **on every compact subset of Ω** .

注. 性质 (2) 即说明了对于一致收敛的函数列, 极限与积分可交换次序, 即

$$\lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} \lim_{n \rightarrow \infty} f_n(z) dz = \int_{\gamma} f(z) dz \quad (5.46)$$

证明.

(1) : Fix $z_0 \in \Omega$, $\forall \epsilon > 0$

Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f , there exists $N \in \mathbb{N}$, s. t.

$$|f(z) - f_n(z)| \leq \epsilon, \quad \forall n \geq N, \forall z \in \Omega \quad (5.47)$$

Since f_N is continuous at z_0 , there exist $\delta > 0$, s. t.

$$|f_N(z) - f_N(z_0)| \leq \epsilon, \quad \forall z \in D_\delta(z_0) \quad (5.48)$$

Therefore

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \quad (5.49)$$

$$\leq 3\epsilon, \quad \forall z \in D_\delta(z_0) \quad (5.50)$$

(2) : Fix $\epsilon > 0$. For any path γ with finite length, $\gamma \subset \Omega$ is a compact subset of Ω .

Let $L = \text{length}(\gamma)$. Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly on compact subset γ , $\exists N \in \mathbb{N}$, s. t.

$$|f_n(z) - f(z)| \leq \frac{\epsilon}{L}, \quad \forall z \in \gamma, \forall n > N \quad (5.51)$$

Hence, for all $n > N$,

$$\left| \int_\gamma f_n(z) dz - \int_\gamma f(z) dz \right| \leq \int_\gamma |f_n(z) - f(z)| \leq \frac{\epsilon}{L} \cdot L = \epsilon, \quad \forall n > N \quad (5.52)$$

Therefore

$$\lim_{n \rightarrow \infty} \int_\gamma f_n(z) dz = \int_\gamma f(z) dz = \int_\gamma \lim_{n \rightarrow \infty} f_n(z) dz \quad (5.53)$$

(3) : 下面分两部分证明. (对应书IP54 Thm 5.3)

- f is holomorphic.

$\forall a \in \Omega$, $\exists r > 0$, s. t. $D_r(a) \subset \Omega$. Let γ be any closed path in $D_r(a)$.

Since f_n is holomorphic on $D_r(a)$, by **Cauchy's Theorem (Thm 4.3.2)**,

$$\int_\gamma f_n(z) dz = 0 \quad (5.54)$$

By (2) proofed previously,

$$\int_\gamma f(z) dz = \lim_{n \rightarrow \infty} \int_\gamma f_n(z) dz = 0 \quad (5.55)$$

¹课堂教材: 《Complex Analysis》 — Elias M. Stein

By **Morera's Theorem (Prop 5.2.2)**, f is holomorphic on $D_r(a)$.

In particular, f is complex differentiable at a . Since a is arbitrary, f is holomorphic on Ω .

- $\{f'_n\}_{n=1}^\infty$ converges uniformly to f' on every compact subset of Ω .

$\forall z_0 \in \Omega, \exists r > 0$, s. t. $\overline{D}_r(z_0) \subset \Omega$.

Since $f_n \rightarrow f$ converges uniformly on $C_r(z_0)$, we have $\forall z \in D_{\frac{r}{2}}(z_0)$

$$\frac{f_n(\zeta)}{(\zeta - z)^2} \rightarrow \frac{f(\zeta)}{(\zeta - z)^2} \text{ uniformly on } C_r(z_0) \quad (5.56)$$

i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$\left| \frac{f_n(\zeta)}{(\zeta - z)^2} - \frac{f(\zeta)}{(\zeta - z)^2} \right| < \frac{\epsilon}{r}, \quad \forall n > N, \forall \zeta \in C_r(z_0) \quad (5.57)$$

Therefore

$$|f'_n(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_{C_r(z_0)} \left(\frac{f_n(\zeta)}{(\zeta - z)^2} - \frac{f(\zeta)}{(\zeta - z)^2} \right) d\zeta \right| \quad (5.58)$$

$$< \frac{1}{2\pi i} \cdot \frac{\epsilon}{r} \cdot 2\pi r i \quad (5.59)$$

$$= \epsilon, \quad \forall n > N, \forall z \in D_{\frac{r}{2}}(z_0) \quad (5.60)$$

It tells $f'_n \rightarrow f'$ uniformly on $D_{\frac{r}{2}}(z_0)$.

For any compact subset $K \subset \Omega$, consider the open covering $\{D_{\frac{1}{2}r_x}(x) \subset \Omega\}_{x \in K}$.

There exists a finite subcovering $K \subset \{D_{r_i}(x_i) \subset \Omega\}_{i=1}^n$.

We can proof $f'_n \rightarrow f'$ uniformly on K .

□

5.5 课堂例题 2024 - 03 - 29

1. Evaluate

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz \quad (5.61)$$

解.

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz = \int_{C_{\frac{1}{3}}(0)} \frac{z}{(z-2)^2} \cdot \frac{1}{z-1} dz + \int_{C_{\frac{1}{3}}(2)} \frac{z}{z-1} \cdot \frac{1}{(z-2)^2} dz \quad (5.62)$$

$$= 2\pi i + 2\pi i \left(\frac{z}{z-1} \right)' \Big|_{z=2} \quad (5.63)$$

$$= 0 \quad (5.64)$$

□

第六章 *Week 6*

6.1 课堂例题 2024 – 04 – 01

本节为习题课. (博士研究生助教代课)

1. 课本第二章练习 $T1, T2, T3, T4$.

6.2 函数项级数，全纯函数解析

回顾 在介绍复数域上函数项级数的性质之前，先回顾一下函数列的性质 (Thm 5.4.1).

定理 6.2.1. 一致收敛 $\Rightarrow$ 积分与极限可交换次序.

Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \rightarrow \mathbb{C}\}_{n=1}^\infty$ be a sequence of continuous functions that converges uniformly to f **on every compact subset of Ω** . Then

(1) f is continuous.

(2) If $\gamma \subset \Omega$ is a path with finite length, then

$$\lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} f(z) dz \quad (6.1)$$

(3) If f_n is holomorphic for all n , then so is f . Moreover,

$\{f'_n\}_{n=1}^\infty$ converges uniformly to f' **on every compact subset of Ω** .

注. • 事实上，结论 (3) 可做推广，即当函数列 $\{f_n\}_{n=1}^\infty$ 满足上述条件时，有：

$\{f_n^{(k)}\}_{n=1}^\infty$ converges uniformly to $f^{(k)}$ **on every compact subset of Ω** .

- 注意实变函数列与复变函数列的**可微性**的区别，即实变函数列不满足定理中的 (3). 下面给出结论 (3) 在实变函数列下的反例.

例 6.2.1. Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $x \in [-1, 1]$. Then $\{f_n\}_{n=1}^\infty$ converges uniformly to $f(x) = |x|$.

Though f_n , $n = 1, 2, \dots$ are differentiable over $[-1, 1]$, the limit function $f(x) = |x|$ is **not differentiable** at $x = 0$.

证明. 详见定理 5.4.1 证明.

□

函数项级数 下面给出函数项级数收敛的定义.

定义 6.2.1. Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \rightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of functions.

We say $\sum_{n=1}^{\infty} f_n$ **converges** if $\{S_N = \sum_{n=1}^N f_n\}_{N=1}^{\infty}$ converges.

We say $\sum_{n=1}^{\infty} f_n$ **converges uniformly** if $\{S_N = \sum_{n=1}^N f_n\}_{N=1}^{\infty}$ converges uniformly).

下面给出判断函数项级数一致收敛性的经典方法 (**Weierstrass M-test**).

命题 6.2.1. Weierstrass M-test.

If $|f_n(z)| \leq M_n$, $n = 1, 2, \dots$, $\forall z \in \Omega$, and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

证明. Let $S_N = \sum_{n=1}^N f_n$, $\forall N \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} M_n < \infty$, then

$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t. $n > m > N$

$$|S_n - S_m| = |f_n(z) + \dots + f_{m+1}(z)| \leq \sum_{j=m+1}^{\infty} M_j \leq \epsilon, \quad \forall z \in \Omega \quad (6.2)$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ converges uniformly. $\Rightarrow \{f_n\}_{n=1}^{\infty}$ converges uniformly. $\square$

解析与全纯等价 下面证明全纯函数均解析 (可展成幂级数).

(该定理与定理 3.1.2 共同说明了, 解析 $\Leftrightarrow$ 全纯)

定理 6.2.2. Suppose f is holomorphic on an open set $\Omega \subset \mathbb{C}$. Then $\forall z_0 \in \Omega$ with $D_r(z_0) \subset \Omega$ for some $r > 0$, f has a power series expansion at z_0 .

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall z \in D_r(z_0) \quad (6.3)$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$, $\forall n \geq 0$.

证明. Fix $z \in D_r(z_0)$. By **CIF (Thm 5.2.1)**,

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (6.4)$$

Then

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \quad (6.5)$$

Since $z \in D_r(z_0)$ is fixed, and $\forall \zeta \in C_r(z_0)$, there exists $0 < r < 1$, s. t.

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < r \quad (6.6)$$

Therefore,

$$\sum_{n=0}^N \left(\frac{z - z_0}{\zeta - z_0} \right)^n \Rightarrow \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}, \quad N \rightarrow \infty, \quad \forall \zeta \in C_r(z_0) \quad (6.7)$$

converges uniformly w.r.t. (with respect to, 关于) $\zeta \in C_r(z_0)$. i.e.

$$\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}, \quad \forall \zeta \in C_r(z_0) \quad (6.8)$$

Let

$$g_N(\zeta) = \sum_{n=0}^N \left(f(\zeta) \cdot \frac{1}{\zeta - z_0} \cdot \left(\frac{z - z_0}{\zeta - z_0} \right)^n \right) \quad (6.9)$$

$$= f(\zeta) \cdot \frac{1}{\zeta - z_0} \sum_{n=0}^N \left(\frac{z - z_0}{\zeta - z_0} \right)^n, \quad \zeta \in C_r(z_0) \quad (6.10)$$

Then we have

$$g_N \Rightarrow f(\zeta) \cdot \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \quad (6.11)$$

$$= \frac{f(\zeta)}{\zeta - z}, \quad N \rightarrow \infty, \quad \zeta \in C_r(z_0) \quad (6.12)$$

converges uniformly w.r.t. $\zeta \in C_r(z_0)$. Therefore, by Thm 6.2.1,

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_r(z_0)} \lim_{N \rightarrow \infty} g_N(\zeta) d\zeta \quad (6.13)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{C_r(z_0)} g_N(\zeta) d\zeta \quad (6.14)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{C_r(z_0)} \left(f(\zeta) \cdot \frac{1}{\zeta - z_0} \sum_{n=0}^N \left(\frac{z - z_0}{\zeta - z_0} \right)^n \right) d\zeta \quad (6.15)$$

$$= \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{2\pi i} \int_{C_r(z_0)} \left(f(\zeta) \cdot \frac{1}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \right) d\zeta \quad (6.16)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \quad (6.17)$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (6.18)$$

By CIF (Thm 5.2.2), we have

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \frac{f^{(n)}(z_0)}{n!}, \quad \forall n \geq 0 \quad (6.19)$$

□

注. 事实上, 该定理也提供了 **CIF 高阶形式 (Thm 5.2.2)** 的另一种证明 (比较系数可得).

6.3 解析延拓

定义 下面先给出解析延拓 (Analytic continuation) 的定义.

定义 6.3.1. Suppose f and F are holomorphic in nonempty regions Ω and $\hat{\Omega}$ respectively with $\Omega \subset \hat{\Omega}$. If $f(z) = F(z)$ in Ω , then we say F is an analytic continuation of f in $\hat{\Omega}$.

解析延拓的唯一性 在说明这之前, 先给出一个有关全纯函数的非常重要的结论.

定理 6.3.1. Suppose f is holomorphic in a region Ω that vanishes on a sequence of distinct points with a limit in Ω . Then $f(z) \equiv 0$ for all $z \in \Omega$.

注. 该定理说明了, 不恒为零的全纯函数的零点均为孤立点 (不为聚点).

证明. 反证法. Suppose $\{w_k\}_{k=1}^{\infty} \subset \Omega$ with $\lim_{k \rightarrow \infty} w_k = z_0 \in \Omega$ and $f(w_k) = 0, k = 1, 2, \dots$.

Since f is holomorphic in $\Omega, \forall z \in D_r(z_0)$

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \text{ for some } r > 0 \quad (6.20)$$

下面分为两步证明.

- We first show $f(z) \equiv 0$ on $D_r(z_0)$.

Assume $f(z) \neq 0$ for $z \in D_r(z_0)$, then $\exists$ smallest integer m s. t. $a_m \neq 0$. Now

$$f(z) = a_m(z - z_0)^m(1 + g(z)), \text{ where } g(z) = \sum_{n=m+1}^{\infty} a_n(z - z_0)^{n-m} \quad (6.21)$$

Since $g(z) \rightarrow 0$ as $z \rightarrow z_0, \forall \epsilon < 1$, there exists $\delta > 0$, s. t.

$$|g(z)| \leq \epsilon < 1, \forall z \in D_{\delta}^*(z_0) \quad (6.22)$$

Since $w_k \rightarrow z_0 \in \Omega, \exists k_0 \in \mathbb{N}$, s. t. $w_{k_0} \in D_{\delta}^*(z_0)$. Then

$$f(w_{k_0}) = a_m(w_{k_0} - z_0)^m(1 + g(w_{k_0})) \neq 0 \quad (6.23)$$

which is a contradiction with that $f(w_{k_0}) = 0$.

- Then we shall show $f \equiv 0$ on Ω .

Let U be the interior of $\{z \in \Omega \mid f(z) = 0\}$. Since $f(z) \equiv 0, \forall z \in D_r(z_0), U \neq \emptyset$ and U is open.

Moreover, 下面我们证明 U is closed.

- For all $\{z_k\}_{k=1}^{\infty} \subset U$ with $z_k \rightarrow p \in \Omega$. 与第一步证明相同, 可以得到
 $\exists r_p > 0$, s. t. $f(z) \equiv 0, \forall z \in D_{r_p}(p)$. 于是 $p \in U$. 即 U 包含了自身序列的所有极限点.
which means that U is closed.

Therefore, $U \subset \Omega$ is both open and closed. Since $U \neq \emptyset$ and Ω is connected, then $U = \Omega$, which means $f \equiv 0$ on Ω .

□

通过上述定理可得到, 全纯函数的取值只由区域上可数个点决定.

推论 6.3.2. Suppose f, g are holomorphic in a region $\hat{\Omega}$ and $f(z) = g(z)$ for all $z \in \Omega$, where Ω is an open subset of $\hat{\Omega}$. Then $f(z) = g(z)$ for $z \in \hat{\Omega}$.

By Cor 6.3.2, 我们得到解析延拓若存在, 则必唯一.

推论 6.3.3. Suppose F and G are both analytic continuation of f into $\hat{\Omega}$. Then

$$F = G \text{ in } \hat{\Omega}$$

6.4 对称原理

引入 在实分析中，我们曾探讨过有关连续函数的延拓 (Tietze 延拓定理).

但在复分析中，对于全纯函数的延拓似乎不再那么容易与显然，因为全纯函数不仅要求在复平面上光滑，而且还具有一些 additional characteristically rigid properties.

(书 P67 Problem 1. 给出了无法 (解析) 延拓至 $\overline{\mathbb{D}}$ 的定义在 $\mathbb{D}$ 上的全纯函数.)

而本节将给出一种十分有用的情况下全纯函数的延拓，即在关于实轴对称的区域上的延拓.

对称原理 为了讨论的方便，接下来的命题将默认以下几个记号：

- Let Ω be an open subset of $\mathbb{C}$ that is **symmetric** w.r.t. the real axis.
- Let Ω^+ denote the part of Ω that lies in the upper half-plane and Ω^- the part that lies in the lower half-plane.

下面给出对称原理.

定理 6.4.1. Symmetric principle.

If f^+ and f^- are holomorphic in Ω^+ and Ω^- respectively that extends continuously to I ($I = \Omega \cap \mathbb{R}$) and $f^+(x) = f^-(x)$ for all $x \in I$. Then

$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases} \quad (6.24)$$

is holomorphic in Ω .

证明. 详见书 P58 Thm 5.5 证明.

□

Schwartz 反射原理 有了上述对称原理的铺垫后，下面给出全纯函数在关于实轴对称区域上的延拓定理. (**Schwartz 反射原理**)

定理 6.4.2. Schwartz reflection principle.

Suppose f is holomorphic in Ω^+ that extends continuously to I and f is real-valued on I . Then $\exists F$ holomorphic in all Ω , s. t. $F = f$ in Ω^+ .

证明. Define $f^-(z) = \overline{f(\bar{z})}$ for $z \in \Omega^-$. Fix $z_0 \in \Omega^-$, $\exists r > 0$, s. t. $D_r(z_0) \subset \Omega^-$.

Then $\bar{z}_0 \in \Omega^+$ and $D_r(\bar{z}_0) \subset \Omega^+$. $\forall x \in D_r(z_0)$, $\bar{x} \in D_r(\bar{z}_0)$

Since f is holomorphic in Ω^+ , by Thm 6.2.2 (全纯函数均解析)

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n, \quad \forall \bar{z} \in D_r(\bar{z}_0) \quad (6.25)$$

Then we have

$$f^-(z) = \sum_{n=0}^{\infty} \bar{a}_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \forall z \in D_r(z_0) \quad (6.26)$$

Therefore, f^- is analytic in ω^- . By Thm 3.1.2, f^- is holomorphic in Ω^- . (解析函数均全纯)

Therefore, by **Symmetric principle** (对称原理),

$$F(z) = \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases} \quad (6.27)$$

is holomorphic in Ω . □

6.5 课堂例题 2024 – 04 – 07

1. (课本 P66 Ex10.)

Weierstrass's theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. **Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z ?**

解. Absolutely No. Take $f(z) = |z|$ continuous on $\mathbb{D}$ into consideration.

If there exists polynomials $\{f_n\}_{n=1}^{\infty}$, s. t. $f_n \Rightarrow f$, then by Thm 6.2.1,

f is holomorphic on $\mathbb{D}$, which is a contradiction with that f is not differentiable at $z = 0$. $\square$

2. Suppose f is entire and real-valued on the real-axis. If $f(1 + i) = 2 + i$, then what is $f(1 - i)$?

解. Ans : $2 - i$. (by **Schwartz reflection principle**) $\square$

3. 课本第二章练习 T7 – 10, T15.

第七章 Week 7

7.1 零点, 极点, 留数

零点 根据定理 6.3.1 知, 不恒为零的全纯函数只含孤立零点. 下面我们将给出非零全纯函数在其孤立零点附近的局部刻画.

下面先给出孤立零点的定义.

定义 7.1.1. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. We say the zero z_0 is isolated if $\exists r > 0$, s. t. $f(z) \neq 0$ for all $z \in D_r^*(z_0)$.

注. By Thm 6.3.1, we note that if $f(z) \not\equiv 0$, $z \in \Omega$ (不全为零), then the zeros of $f(z)$ are isolated.

下面给出非零全纯函数在其孤立零点附近的局部刻画.

定理 7.1.1. Suppose $f(z) \not\equiv 0$ is holomorphic in a region Ω . z_0 is a zero of f . Then $\exists r > 0$ and **nonvanishing** holomorphic function $g(z)$ in $D_r(z_0)$ and a **unique** integer n , s. t.

$$f(z) = (z - z_0)^n g(z), \quad z \in D_r(z_0)$$

注. • $f(z) \not\equiv 0$ 指的是 f 不恒为零, 而 g nonvanishing 指的是 g 恒不为零.

- In this theorem, we say z_0 is a zero of f of multiplicity of n .

If $n = 1$, we say z_0 is a simple zero of f .

证明.

- **存在性:** Since f is holomorphic, by Thm 6.2.2, $\exists R > 0$, s. t.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad \forall z \in D_R(z_0) \quad (7.1)$$

$f(z_0) = 0 \Rightarrow a_0 = 0$. Since $f(z) \neq 0$, $\exists$ the smallest integer n , s. t. $a_n \neq 0$. Then

$$f(z) = (z - z_0)^n(a_n + a_{n+1}(z - z_0) + \cdots) = (z - z_0)^n g(z) \quad (7.2)$$

Clearly, $\exists 0 < r < R$, s. t. $g(z) \neq 0$ for all $z \in D_r(z_0)$.

- 唯一性: 详见书 P73 Thm 1.1 证明.

□

极点 下面给出复变函数的极点的定义.

定义 7.1.2. We say $f : D_r^*(z_0) \rightarrow \mathbb{C}$ has a **pole** at z_0 if $\frac{1}{f}$ is holomorphic in $D_r(z_0)$ and has a zero at z_0 .

注. • 牢林此处的定义并不严谨, 并未对 $1/f$ 在 z_0 处无定义的情况说明 **pole** 的定义. 事实上这会导致后面对奇点附近性质的讨论带来不便, 下面引用书 P74 的定义.

定义 7.1.3. We say $f : D_r^*(z_0) \rightarrow \mathbb{C}$ has a **pole** at z_0 , if the function $\frac{1}{f}$, defined to be zero at z_0 , is holomorphic in a full neighbourhood of z_0 .

- 由定义可知, a pole of a function is isolated.

根据非零全纯函数在孤立零点附近的局部刻画 (Thm 7.1.1), 可以很容易得到全纯函数在其极点附近的局部刻画.

定理 7.1.2. If f has a pole at z_0 , then $\exists r > 0$ and a **nonvanishing** holomorphic function $h(z)$ in $D_r(z_0)$ and a **unique** positive integer n , s. t.

$$f(z) = (z - z_0)^{-n} h(z), \quad z \in D_r^*(z_0)$$

证明. 详见书 P74 Thm 1.2 证明.

□

留数 在定理 7.1.2 的基础上, 可以更进一步给出更精细的刻画. 对于 n 阶极点, 我们存在这样的刻画.

定理 7.1.3. If f has a pole of order n at z_0 , then we can write

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z) \quad (7.3)$$

where $G(z)$ is holomorphic in some neighbourhood of z_0 .

注. • The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} \quad (7.4)$$

is called the principal part (or singular part) of f at the pole z_0 .

- $G(z)$ is called the holomorphic part of f at the pole z_0 .
- The coefficient a_{-1} is called the residue of f at the pole z_0 . We write $\text{Res}_{z_0} f = a_{-1}$.

证明. 详见书 P75 Thm 1.3 证明. □

关于**留数**的用途和含义, 在于其绕对应极点的环路积分之中.

对于 $f : \Omega \rightarrow \mathbb{C}$ with a pole $z_0 \in \Omega$, since $\frac{a_{-k}}{(z-z_0)^k}, k = 2, \dots, n$ have primitives and G is holomorphic, we have

$$\int_{C_r(z_0)} f(z) dz = \int_{C_r(z_0)} \frac{a_{-1}}{z-z_0} dz = 2\pi i \cdot a_{-1} \quad (7.5)$$

环路积分的值只剩下与 a_{-1} 有关, 此即为 “**留数**” 之意.

下面介绍留数的**计算技巧**. 设 z_0 为 f 的 n 阶极点.

- $n = 1$ 时, $\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} (z - z_0) f(z)$
- $n > 1$ 时, 我们有

$$\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z) \quad (7.6)$$

(根据定理 7.1.3 的公式可轻松得证.)

7.2 Laurent Series Expansion

事实上, 对于全纯函数 f , 其不仅能在定义域内展开为幂级数 (Thm 6.2.2), 其同样能在极点周围类似地展开为幂级数的形式, 此即为 **Laurent Series Expansion** (洛朗级数展开).

定理 7.2.1. Let f be holomorphic on a region containing the annulus and its boundary

$$\mathcal{A} = \{z \mid r_1 < |z - z_0| < r_2\}, \text{ where } 0 \leq r_1 < r_2 \quad (7.7)$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad (7.8)$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \text{ for any } r \in [r_1, r_2] \quad (7.9)$$

注. • The series in the Theorem is called the Laurent Series Expansion of f near z_0 or in the annulus.

• 在同一圆环域内, Laurent 展式唯一; 在不同的圆环域内, Laurent 展式可能不同.

证明. Fix $z \in \mathcal{A}$, $\exists \delta > 0$, s. t. $C_\delta(z) \subset \mathcal{A}$.

Consider

$$g(\zeta) = \frac{f(\zeta)}{\zeta - z} \quad (7.10)$$

Then $g(\zeta)$ is holomorphic in a region containing $\mathcal{A} \setminus D_\delta(z)$ and its boundary.

By the **principle of contour deformation** (Thm 5.1.1, 闭路变形原理)

$$\int_{C_{r_2}(z_0)} g(\zeta) d\zeta = \int_{C_\delta(z)} g(\zeta) d\zeta + \int_{C_{r_1}(z_0)} f(\zeta) d\zeta \quad (7.11)$$

By **CIF** (Thm 5.2.1)

$$2\pi i \cdot f(z) = \int_{C_\delta(z)} g(\zeta) d\zeta \quad (7.12)$$

Thus

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (7.13)$$

下面分别计算积分 $\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$ 和 $-\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$.

- If $\zeta \in C_{r_2}(z_0)$, then $|\zeta - z_0| > |z - z_0|$.

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \quad (7.14)$$

converges w.r.t. ζ . Hence

$$\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \quad (7.15)$$

(此处具体证明过程可见定理 6.2.2 的证明.)

- If $\zeta \in C_{r_1}(z_0)$, then $|\zeta - z_0| < |z - z_0|$.

$$-\frac{1}{\zeta - z} = \frac{1}{z - \zeta} = \frac{1}{(z - z_0) - (\zeta - z_0)} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^n \quad (7.16)$$

$$= \sum_{n=0}^{\infty} \frac{(\zeta - z_0)^{n-1}}{(z - z_0)^n} \quad (7.17)$$

$$= \sum_{n=-1}^{-\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \quad (7.18)$$

Hence

$$-\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=-1}^{-\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \quad (7.19)$$

由于 $\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$, $\forall n$ 在 $\mathcal{A}$ 上 holomorphic, 因此根据闭路变形原理 (Thm 5.1.1), $\forall r \in [r_1, r_2]$

$$\int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (7.20)$$

$$\int_{C_{r_1}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (7.21)$$

Therefore

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (7.22)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n + \sum_{n=-1}^{-\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \quad (7.23)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n + \sum_{n=-1}^{-\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \quad (7.24)$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \quad (7.25)$$

$$= \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \text{ where } a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \text{ for any } r \in [r_1, r_2] \quad (7.26)$$

□

7.3 课堂例题 2024 - 04 - 08

1. Find the Laurent Expansion of

$$\frac{1}{z^2(z-i)} \text{ in } \frac{1}{4} < |z-i| < \frac{3}{4} \quad (7.27)$$

解. Ans:

$$\sum_{n=-1}^{\infty} (n+2)i^{n+1}(z-i)^n \quad (7.28)$$

□

2. Find the Laurent Expansion of

$$\frac{z^3}{1+z^2} \text{ in } 2 < |z| < 4 \quad (7.29)$$

解. Ans:

$$= \frac{z}{1+\frac{1}{z^2}} = z \cdot \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n \quad (7.30)$$

□

3. Find the Laurent Expansion of

$$f(z) = \frac{z^2 - 2z + 5}{(z-2)(z^2+1)} \quad (7.31)$$

in $1 < |z| < 2$, $2 < |z| < +\infty$ respectively.

解. $f(z) = \frac{1}{z-2} - \frac{2}{z^2+1}$.

• In the annulus $1 < |z| < 2$,

$$f(z) = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} - \frac{2}{z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - \frac{2}{z^2} \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n \quad (7.32)$$

$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{z^{2n}} \quad (7.33)$$

• In the annulus $2 < |z| < +\infty$,

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} - \frac{2}{z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=1}^{\infty} \frac{2(-1)^n}{z^{2n}} \quad (7.34)$$

□

7.4 Residue Formula

引入 对于单连通区域, **Cauchy's Theorem (Thm 4.3.2)** 已经告诉了我们全纯函数的环路积分为 0.

但对于更一般的区域, 若其中含有极点, 则 **Cauchy's Theorem** 便不再奏效. 此时便需要使用接下来所要介绍的 **Residue Formula** 来进行计算.

Residue Formula 下面先给出单个极点的圆形环路上函数的积分值.

定理 7.4.1. Suppose f is holomorphic in a region containing $\overline{D_r^*(z_0)}$, $r > 0$, and z_0 is a pole of f . Then

$$\int_{C_r(z_0)} f(z) dz = 2\pi i \cdot \text{Res}_{z_0} f \quad (7.35)$$

证明. 证明是 trivial 的. Suppose z_0 is a pole of order n . Then by **Thm 7.1.3**,

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z) \quad (7.36)$$

where $G(z)$ is holomorphic in $\overline{D_r(z_0)}$.

Since $\frac{a_{-k}}{(z - z_0)^k}$, $k = 2, 3, \dots, n$ admit primitives and G is holomorphic in a region containing $\overline{D_r^*(z_0)}$, this yields

$$\int_{C_r(z_0)} f(z) dz = \int_{C_r(z_0)} \frac{a_{-1}}{z - z_0} dz = 2\pi i \cdot a_{-1} = 2\pi i \cdot \text{Res}_{z_0} f \quad (7.37)$$

□

下面给出 **Residue Formula**. 它给出了环路内部存在有限个极点时的积分计算公式.

定理 7.4.2. Residue Formula.

Suppose f is holomorphic in an open set containing a contour γ and its interior except for poles $z_1, \dots, z_n \in \text{Interior}(\gamma)$. Then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z_k} f \quad (7.38)$$

证明. By **Principle of Contour Deformation (Cor 5.1.3, 闭路变形原理)**,

$$\int_{\gamma} f(z) dz = \sum_{k=1}^n \int_{C_{r_k}(z_k)} f(z) dz \quad (7.39)$$

where $C_{r_k}(z_k)$, $k = 1 \sim n$ are disjoint circles in $\text{Interior}(\gamma)$.

Then by **Thm 7.4.1**, the desired result follows. □

7.5 课堂例题 2024 - 04 - 12

1. (课本 P103 T2.)

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx \quad (7.40)$$

解. Consider $f(z) = \frac{1}{1+z^4}$ and the contour $\gamma_1 \circ \gamma_2$.

(γ_1 为 $(-R, 0)$ 到 $(R, 0)$ 的实直线, γ_2 为 $(R, 0)$ 到 $(-R, 0)$ 的上半圆周)

For sufficiently large R , the contour $\gamma_1 \circ \gamma_2$ contains poles $e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}$ of f .

By the **Residue Formula (Thm 7.4.2)**,

$$\int_{\gamma_1 \circ \gamma_2} f(z) dz = 2\pi i \left(\text{Res}_{e^{\frac{\pi i}{4}}} f + \text{Res}_{e^{\frac{3\pi i}{4}}} f \right) \quad (7.41)$$

By the **L'Hospital's Rule (Thm A.1.1)**, since $e^{\frac{\pi i}{4}}$ is a simple pole of f , we compute

$$\text{Res}_{e^{\frac{\pi i}{4}}} f = \lim_{z \rightarrow e^{\frac{\pi i}{4}}} (z - e^{\frac{\pi i}{4}}) f(z) = \lim_{z \rightarrow e^{\frac{\pi i}{4}}} \frac{z - e^{\frac{\pi i}{4}}}{1 + z^4} \stackrel{\text{L'Hospital}}{=} \frac{1}{4e^{\frac{3\pi i}{4}}} = \frac{1}{4} e^{-\frac{3\pi i}{4}} \quad (7.42)$$

Similarly, we have

$$\text{Res}_{e^{\frac{3\pi i}{4}}} f = \lim_{z \rightarrow e^{\frac{3\pi i}{4}}} (z - e^{\frac{3\pi i}{4}}) f(z) = \frac{1}{4} e^{-\frac{\pi i}{4}} \quad (7.43)$$

Then

$$\int_{\gamma_1 \circ \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz \quad (7.44)$$

$$= 2\pi i \left(\text{Res}_{e^{\frac{\pi i}{4}}} f + \text{Res}_{e^{\frac{3\pi i}{4}}} f \right) = \frac{\sqrt{2}}{2} \pi \quad (7.45)$$

Note that

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_{\gamma_2} \frac{1}{1+z^4} dz \right| \leq \sup_{z \in \gamma_2} |f(z)| \cdot \text{length}(\gamma_2) \quad (7.46)$$

Since $|1+z^4| \geq |z^4| - 1$, then $\sup_{z \in \gamma_2} \leq \frac{1}{R^4-1}$.

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \sup_{z \in \gamma_2} |f(z)| \cdot \text{length}(\gamma_2) \leq \frac{\pi R}{R^4-1} \rightarrow 0, \text{ as } R \rightarrow \infty \quad (7.47)$$

Thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \lim_{R \rightarrow \infty} \int_{\gamma_1} f(z) dz = \lim_{R \rightarrow \infty} \int_{\gamma_1} f(z) dz + \lim_{R \rightarrow \infty} \int_{\gamma_2} f(z) dz \quad (7.48)$$

$$= \lim_{R \rightarrow \infty} \int_{\gamma_1 \circ \gamma_2} f(z) dz \quad (7.49)$$

$$= \frac{\sqrt{2}}{2} \pi \quad (7.50)$$

□

2. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^n} dx, \quad n \geq 2 \quad (7.51)$$

3. (课本 P103 T3.)

Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx, \quad a > 0 \quad (7.52)$$

解. Let $f(z) = \frac{e^{iz}}{z^2 + a^2}$. Consider the contour $\gamma_1 \circ \gamma_2$.

(γ_1 为 $(-R, 0)$ 到 $(R, 0)$ 的实直线, γ_2 为 $(R, 0)$ 到 $(-R, 0)$ 的上半圆周)

Since $\frac{\sin x}{x^2 + a^2}$ 为奇函数,

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x^2 + a^2} dx = 0 \quad (7.53)$$

Then

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + a^2} dx = \lim_{R \rightarrow \infty} \int_{\gamma_1} f(z) dz \quad (7.54)$$

By the **Residue Formula (Thm 7.4.2)**,

$$\int_{\gamma_1 \circ \gamma_2} f(z) dz = 2\pi i \cdot \text{Res}_{ai} f = 2\pi i \lim_{z \rightarrow ai} \frac{e^{iz}}{z + ai} = \frac{\pi e^{-a}}{a} \quad (7.55)$$

Since

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \sup_{0 \leq \theta \leq \pi} \left| \frac{e^{i(R \cos \theta + iR \sin \theta)}}{R^2 e^{i2\theta} + a^2} \right| \cdot \pi R \leq \frac{\pi R}{R^2 - a^2} \rightarrow 0, \quad \text{as } R \rightarrow \infty \quad (7.56)$$

Therefore

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + a^2} dx = \lim_{R \rightarrow \infty} \int_{\gamma_1} f(z) dz = \lim_{R \rightarrow \infty} \int_{\gamma_1 \circ \gamma_2} f(z) dz \quad (7.57)$$

$$= \frac{\pi e^{-a}}{a} \quad (7.58)$$

□

4. 课本第三章练习 T1 ~ T8.

5. 课本 P79 例 2.

第八章 Week 8

8.1 均值定理

下面补充一个 CIF (Thm 5.2.1, 柯西积分公式) 的推论, 即均值定理.

命题 8.1.1. (Mean Value Property.)

If f is holomorphic on $D_R(z_0)$, where $z_0 \in \mathbb{C}$, $R > 0$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta, \quad 0 < r < R \quad (8.1)$$

证明. By CIF (Thm 5.2.1),

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} r \cdot i \cdot e^{i\theta} d\theta \quad (8.2)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \quad (8.3)$$

□

8.2 奇点

下面给出奇点的定义.

定义 8.2.1. A complex number z_0 is a singular point (or a singularity) of f if f is not analytic at z_0 . We say z_0 is an isolated singularity if f is analytic in a deleted neighbourhood of z_0 .

注. 大多数情况下我们研究的都是孤立奇点, 但也存在着非孤立奇点.

例 8.2.1. • 0 is an isolated singularity of $\frac{1}{z}, \frac{1}{\sin z}, \frac{1}{e^z-1}$.

• **Poles are isolated singularities.** (极点均为孤立奇点)

• For

$$\frac{1}{\sin \frac{\pi}{z}} \quad (8.4)$$

0 is not an isolated singularity.

8.2.1 Classification of isolated singularities

下面我们对奇点进行分类.

定义 8.2.2. Let $f : D_r^*(z_0) \rightarrow \mathbb{C}$ where $r > 0, z_0 \in \mathbb{C}$ be holomorphic with the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \quad (8.5)$$

- (1) z_0 is called a removable singularity (可去奇点) if $a_{-n} = 0, n = 1, 2, \dots$.
- (2) z_0 is called a pole (极点) if $a_{-n} \neq 0, a_{-(n+k)} = 0, k = 1, 2, \dots$.
- (3) z_0 is called a essential singularity (本性奇点) if $a_{-n} \neq 0$ for infinitely many $n \geq 1$.

例 8.2.2. • $f(z) = z^{-n}, n \geq 1$ has a **pole** of order n at $z = 0$.

• $f(z) = e^{\frac{1}{z}}$ has an **essential singularity** at $z = 0$.

• $f(z) = \frac{e^z-1}{z}$ has a **removable singularity** at $z = 0$.

证明.

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} \quad (8.6)$$

□

8.3 孤立奇点的等价刻画

下面主要研究孤立奇点附近的性态，并给出各类孤立奇点的等价刻画.

8.3.1 Removable Singularity

首先，介于可去奇点的良好性质，我们可在简单的操作后令函数全纯，即：

注. If z_0 is a removable singularity of $f : D_r^*(z_0) \rightarrow \mathbb{C}$, then we define $f(z_0) = a_0$ so that f is holomorphic in $D_r(z_0)$, where

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D_r^*(z_0) \quad (8.7)$$

下面给出可去奇点的等价刻画.

定理 8.3.1. If z_0 is a singularity of f , then

$$z_0 \in \mathbb{C} \text{ is a removable singularity} \quad \text{iff} \quad f \text{ is bounded near } z_0. \\ \text{(i.e. in a deleted neighbourhood of } z_0)$$

证明.

“ $\Rightarrow$ ” : We can define the value of f at z_0 s. t.

$$f \text{ is holomorphic in } D_r(z_0) \text{ for some } r > 0.$$

In particular, f is continuous in $\overline{D_{\frac{r}{2}}(z_0)}$ and so f is bounded near z_0 .

“ $\Leftarrow$ ” : Define

$$g : D_r(z_0) \longrightarrow \mathbb{C} \quad (8.8)$$

$$z \longmapsto g(z) = \begin{cases} (z - z_0)^2 f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases} \quad (8.9)$$

Then

$$g'(z) = 2(z - z_0)f(z) + (z - z_0)^2 f'(z), \quad \forall z \in D_r^*(z_0) \quad (8.10)$$

Since f is bounded near z_0 ,

$$\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} (z - z_0)f(z) = 0 \quad (8.11)$$

Thus g is holomorphic in $D_r(z_0)$ and so g is analytic in $D_r(z_0)$,

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad \forall z \in D_r(z_0) \quad (8.12)$$

Note that $c_0 = g(z_0) = 0$, $c_1 = g'(z_0) = 0$. Thus

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+2}, \quad \text{where } a_n = c_{n+2}, n = 0, 1, 2, \dots \quad (8.13)$$

Therefore

$$f(z) = (z - z_0)^{-2} g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall z \in D_r^*(z_0) \quad (8.14)$$

and so z_0 is a removable singularity of f .

□

从上述定理的证明过程中，可以直接得到下面的推论，也是对可去奇点的等价刻画.

推论 8.3.2. If z_0 is a singularity of f , then

$$z_0 \text{ is a removable singularity} \quad \text{iff} \quad \lim_{z \rightarrow z_0} (z - z_0)f(z) = 0.$$

8.3.2 Pole

作为定理 8.3.1 的推论，下面我们给出极点的等价刻画.

推论 8.3.3. ¹ If z_0 is a singularity of f , then

$$z_0 \text{ is a pole} \quad \text{iff} \quad |f(z)| \rightarrow \infty \text{ as } z \rightarrow z_0.$$

证明. “ $\Rightarrow$ ”: $\frac{1}{f(z_0)} = 0 \Rightarrow |f(z)| \rightarrow \infty$ as $z \rightarrow z_0$.

“ $\Leftarrow$ ”: Suppose $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$, then $\exists r > 0$, s. t.

$$f(z) \neq 0, \quad \forall z \in D_r^*(z_0)$$

and so $\frac{1}{f}$ is holomorphic on $D_r^*(z_0)$. Moreover, $\left|\frac{1}{f(z)}\right| \rightarrow 0$ as $z \rightarrow z_0$.

By **Thm 8.3.1**, $\frac{1}{f}$ is bounded near z_0 ,

$$\Rightarrow z_0 \text{ is a removable singularity of } \frac{1}{f}.$$

Since $\left|\frac{1}{f(z)}\right| \rightarrow 0$ as $z \rightarrow z_0$, we have

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \forall z \in D_r^*(z_0), \text{ where } a_0 = 0 \quad (8.15)$$

Therefore, if we define $\frac{1}{f(z_0)} = 0$, then $\frac{1}{f}$ is holomorphic on $D_r(z_0)$. By **Def 7.1.3**, z_0 is a pole of f . $\square$

8.3.3 Essential Singularity

在排除了定理 8.3.1 和 **Cor cor 8.3.3** 的情况后，下面我们给出本性奇点的等价刻画.

推论 8.3.4. If z_0 is a singularity of f , then

$$z_0 \text{ is an essential singularity} \quad \text{iff} \quad \lim_{z \rightarrow z_0} |f(z)| \text{ does not exists.}$$

(Here we allow the limit to be ∞)

例 8.3.1. Consider $f(z) = e^{\frac{1}{z}}$. Since

$$\lim_{\substack{z \in \mathbb{R} \\ z \rightarrow 0^+}} e^{\frac{1}{z}} = \infty, \quad \lim_{\substack{z \in \mathbb{R} \\ z \rightarrow 0^-}} e^{\frac{1}{z}} = 0 \quad (8.16)$$

Therefore 0 is an essential singularity of f .

¹对应课本 P85 Cor 3.2

8.4 课堂例题 2024 - 04 - 15

1. (课前 Question 1.)

Let $z_1, z_2 \in \mathbb{C}$ with $\operatorname{Re} z_1 \leq 0, \operatorname{Re} z_2 \leq 0$. Show

$$|e^{z_1} - e^{z_2}| \leq |z_1 - z_2|$$

2. (课前 Question 2.)

If f, g are entire functions that agree in infinite number of points, then $f = g$?

解. Eg: $f(z) = \sin z, g(z) = e^1 \sin z$.

□

3. (课前 Question 3.)

Is there a holomorphic function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ with a simple pole at $z = 0$, s. t.

$$\int_{C_1(0)} f(z) dz = 0 ? \quad (8.17)$$

解. simple pole $\Rightarrow \operatorname{Res}_0 f \neq 0 \Rightarrow \int_{C_1(0)} f(z) dz = 2\pi i \cdot \operatorname{Res}_0 f \neq 0$.

□

4. 课本第三章练习 T13.

8.5 本性奇点附近的不稳定性

由定理 8.3.1 和推论 8.3.3 可知, 函数在可去奇点和极点附近的性态是“稳定的”, 是“有迹可循的”. 但接下来我们将说明, 全纯函数在本性奇点附近的性态是“不稳定的” (erratically), 是“无迹可寻的”.

8.5.1 Casorati – Weierstrass 定理

下面的定理说明了, 全纯函数在本性奇点附近的值域是整个复数域 $\mathbb{C}$ 的稠密子集.

定理 8.5.1.² **Casorati-Weierstrass.**

Let $f : D_r^*(z_0) \rightarrow \mathbb{C}$ be holomorphic and z_0 be an essential singularity of f . Then the image of f is dense in $\mathbb{C}$, that is,

$$\forall w \in \mathbb{C}, \forall \epsilon > 0, \exists z \in D_r^*(z_0), \text{ s. t. } |f(z) - w| < \epsilon.$$

证明. 反证法. Assume $\exists w \in \mathbb{C}, \delta > 0, \text{ s. t. } \forall z \in D_r^*(z_0), |f(z) - w| \geq \delta$.

Define $g(z) = \frac{1}{f(z) - w}$ on $D_r^*(z_0)$. Then $g(z)$ is holomorphic and bounded on $D_r^*(z_0)$.

By **Thm 8.3.1**, z_0 is a removable singularity of g .

Define $g(z_0)$ s. t. $g(z)$ is holomorphic on $D_r(z_0)$.

- If $g(z_0) \neq 0$, since

$$f(z) = \frac{1}{g(z)} + w \tag{8.18}$$

Then z_0 is a removable singularity of f , which is a contradiction.

- If $g(z_0) = 0$, then z_0 is a pole of $f(z) - w$.

By **Cor 8.3.3**, z_0 is also a pole of $f(z)$, which is also a contradiction.

Therefore, the proof is complete. □

² 《Complex Analysis》 — Elias M. Stein P86 Thm 3.3

8.5.2 *Picard* 定理

下面给出一个有关本性奇点附近性态更强的结论，即 **Picard 定理**. 它说明了在至多除去一个点后，全纯函数在本性奇点附近能将复数域上每个点取无穷多次.

定理 8.5.2. **Picard.**

If z_0 is an essential singularity of f , then $f(z)$ can take any complex number infinitely many times with at most one exception.

8.6 亚纯函数，扩充复平面，无穷远处孤立奇点

8.6.1 亚纯函数

在给出了孤立奇点的定义和刻画后，下面来讨论一类比全纯条件稍弱的亚纯函数.

定义 8.6.1. A function f on an open set Ω is meromorphic if f is holomorphic on Ω except for a sequence of poles in Ω and the sequence has no limit in Ω .

- 注.**
- 条件 “a sequence of poles” 既可为有限序列也可为无穷序列.
 - 条件 “the sequence has no limit in Ω ” 说明极点序列在 Ω 中无聚点.

例 8.6.1. 下面给出几个亚纯函数的例子.

- $f(z) = \frac{1}{\sin z}$ is a meromorphic function on $\mathbb{C}$.

证明. $\sin z = 0 \Leftrightarrow z = k\pi, k \in \mathbb{Z}$

□

•

$$g(z) = \frac{1}{\sin \frac{1}{1-z}} \quad (8.19)$$

is a meromorphic function on $\mathbb{D}$. However, $\forall \epsilon > 0$, $g(z)$ is not meromorphic on $\mathbb{D}_{1+\epsilon}(0)$.

证明. Poles of $g(z)$ are $1 - \frac{1}{k\pi}, k = 1, 2, \dots$.

□

8.6.2 扩充复平面

与扩充实数系概念类似，下面给出扩充复平面的定义.

定义 8.6.2. The extended complex plane

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

注. 由拓扑知识可知，对 $\mathbb{C}$ 添加元素 ∞ 后进行单点紧致化³后，可得到新的拓扑空间与球面 S^2 同胚，即

$$\hat{\mathbb{C}} \cong S^2$$

此时可将扩充复平面 $\hat{\mathbb{C}}$ 视作一个球面，其中北极点为无穷远点 ∞ .

8.6.3 无穷远处孤立奇点

定义 首先给出无穷远处孤立奇点的定义.

定义 8.6.3. Suppose $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a function on $\hat{\mathbb{C}}$. We say ∞ is an isolated singularity of f if $\exists R > 0$, s. t.

f is holomorphic on $\{w \in \mathbb{C} \mid |w| > R\}$.

The set $\{w \in \mathbb{C} \mid |w| > R\}$ for some $R > 0$ is called a deleted neighbourhood of ∞ .

- 注.**
- 此处若将扩充复平面 $\hat{\mathbb{C}}$ 视作球面，则有关无穷远处去心邻域的定义是自然的.
 - 与数学分析处理方法类似， f 在 ∞ 附近的性质与函数 $F(z) = f(\frac{1}{z})$ 在 $z = 0$ 处性质等价.

³可参考书籍《基础拓扑学讲义》— 尤承业 P60 习题 18、19

下面利用 $F(z) = f(\frac{1}{z})$ 的转化, 我们给出无穷远处可去奇点、极点、本性奇点的定义.

定义 8.6.4. Suppose $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be a function on $\hat{\mathbb{C}}$. We say that

- f has a removable singularity at ∞ if F has a removable singularity at 0.
- f has a pole at ∞ if F has a pole at 0.
- f has an essential singularity at ∞ if F has an essential singularity at 0.

例 8.6.2. 下面给出几个无穷远处孤立奇点的例子.

- $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, where $a_k \in \mathbb{C}$, has a pole of order n at ∞ .

证明. Consider $F(z) = p(\frac{1}{z}) = \frac{1}{z^n} + \cdots + \frac{a_{n-1}}{z} + a_n$. Then $F(z)$ has a pole of order n at 0. $\square$

- $e^z, \sin z, \cos z$ all have essential singularities at ∞ .

无穷远处孤立奇点的等价刻画 下面给出无穷远处孤立奇点的等价刻画.

命题 8.6.1. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function. Then

- (1) f has a pole at ∞ iff f is a nonconstant polynomial.
- (2) f has a removable singularity at ∞ iff f is a constant.

证明. Since f is entire, its Taylor Expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in \mathbb{C}$.

By definition,

- (1) f has a pole at ∞ iff $F(z) = f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n}$ has a pole at 0,
which is equivalent to the existence of a positive integer N s. t. $a_N \neq 0, a_{N+k} = 0, k = 1, 2, \dots$.
Then $f(z) = F(\frac{1}{z}) = \sum_{n=0}^N a_n z^n$ is a nonconstant polynomial.

- (2) f has a removable singularity at ∞ iff $F(z) = f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n}$ has a removable singularity at 0,
which is equivalent to $a_k = 0, k = 1, 2, \dots$.

Then $f(z) = F(\frac{1}{z}) = a_0$ is constant.

$\square$

8.7 课堂例题 2024 – 04 – 19

1. 课本第三章练习 $T14$ 、 $T15$.

第九章 Week 9

9.1 扩充复平面 $\hat{\mathbb{C}}$ 上亚纯函数的刻画

下面我们将说明, 定义在扩充复平面 $\hat{\mathbb{C}}$ 上的亚纯函数都是有理函数. 在此之前, 先来回顾一下有理函数的定义.

定义 9.1.1. A rational function in z is a quotient of polynomials in z .

这里再来补充一下扩充复平面 $\mathbb{C}$ 上亚纯函数的定义. (书P87, 牢林这里又漏了定义!)

定义 9.1.2. A meromorphic function in the complex plane that is either holomorphic at ∞ or has a pole at ∞ is said to be meromorphic in the extended complex plane.

下面给出亚纯函数在扩充复平面 $\hat{\mathbb{C}}$ 上的刻画.

定理 9.1.1. The meromorphic functions on $\hat{\mathbb{C}}$ are the rational functions.

证明. Let $f : \hat{\mathbb{C}} \rightarrow \mathbb{C}$ be a meromorphic function on $\hat{\mathbb{C}}$. Whether f has a pole or removable singularity at ∞ , f is holomorphic in the set $\{z \in \mathbb{C} \mid |z| > R\}$ for some R .

Thus all other possible poles in $\mathbb{C}$ lie in the compact $\{z \in \mathbb{C} \mid |z| \leq R\}$, and so f can have only finitely many poles in $\mathbb{C}$, say $z_1, \dots, z_k$.²

¹ 《Complex Analysis》— Elias M. Stein

²By **Bolzano-Weierstrass Theorem**, 任一有界数列必有收敛子列 (高维仍成立), 若 poles infinite, 则在紧集 $\{z \in \mathbb{C} \mid |z| \leq R\}$ 中有聚点, 与 meromorphic 矛盾.

Hence there are positive numbers $n_1, \dots, n_k$, s. t.

$$G(z) = (z - z_1)^{n_1} \cdots (z - z_k)^{n_k} f(z) \text{ has only removable singularities in } \mathbb{C}.$$

After filling the definition of G at $z_1, \dots, z_k$, we may assume $G(z)$ is entire. Observe that

$$G(z) \text{ is rational} \quad \text{iff} \quad f(z) \text{ is rational.}$$

For $G(z)$ at ∞ , there are 3 possibilities:

- (1) If $G(z)$ has a pole at ∞ , then by **Prop 8.6.1**, $G(z)$ is a nonconstant polynomial.
- (2) If $G(z)$ has a removable singularity at ∞ , then $G(z)$ is a constant.
- (3) Assume ∞ is an essential singularity of $G(z)$. This is impossible for the Laurent expansion of

$$\tilde{G}(z) := G\left(\frac{1}{z}\right) = \left(\frac{1}{z} - z_1\right)^{n_1} \cdots \left(\frac{1}{z} - z_k\right)^{n_k} f\left(\frac{1}{z}\right) \text{ near the origin} \quad (9.1)$$

has only finitely negative powers. Thus ∞ is not an essential singularity of $G(z)$.

Therefore, $G(z)$ is rational, then $f(z)$ is rational. □

9.2 辐角原理, Rouché's Theorem

9.2.1 The Argument Principle

下面给出辐角原理的叙述.

定理 9.2.1. Argument Principle.

Suppose f is meromorphic in an open set containing a contour γ and its interior. If f has no poles and never vanishes on γ , then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \#(\text{zeros of } f \text{ in Interior}(\gamma)) - \#(\text{poles of } f \text{ in Interior}(\gamma)) \quad (9.2)$$

where the zeros and poles are counted with their multiplicities (i.e. orders).

注. 可从几何视角来看待辐角原理.

函数 f 实际上引导了一个复平面 z -plane 到另一个复平面 w -plane 的变换, 即

The mapping $w = f(z)$ maps the contour γ to a closed path Γ in w -plane.

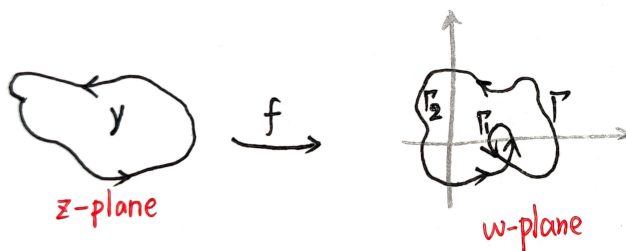


图 9.1: mapping $w = f(z)$

Let $\Gamma(t) = (f \circ \gamma)(t) = f(\gamma(t))$, $t \in [0, 1]$. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{dw}{w} \quad (9.3)$$

闭曲线 Γ 可拆为若干简单闭曲线, 即

$$\Gamma = \Gamma_1 + \cdots + \Gamma_k$$

- 若 Γ_i 内部无原点, 则 $\frac{1}{w}$ holomorphic, $\int_{\Gamma_i} \frac{dw}{w} = 0$.
- 若 Γ_i 内部有原点, 则 $\int_{\Gamma_i} \frac{dw}{w} = 2\pi i$.

从而

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{dw}{w} = \Gamma \text{ 绕原点圈数} \quad (9.4)$$

证明. Let $f \not\equiv 0$ (不恒为 0) be a meromorphic function on an open set Ω .

- If f has a pole of order n at $a \in \Omega$, then by **Thm 7.1.2**, $\exists$ a nonvanishing (恒不为 0) holomorphic function $h(z)$ in a neighbourhood of a , s. t.

$$f(z) = (z - a)^{-n} h(z)$$

Then

$$\frac{f'(z)}{f(z)} = -\frac{n}{z - a} + \frac{h'(z)}{h(z)} \quad (9.5)$$

$$\text{Res}_a \frac{f'}{f} = -n = - \text{the order of the pole } a \text{ of } f \quad (9.6)$$

- If $\beta \in \Omega$ is a zero of order m of f , then by **Thm 7.1.1**, $\exists$ a nonvanishing holomorphic function $g(z)$ in a neighbourhood of β , s. t.

$$f(z) = (z - \beta)^m g(z)$$

Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z - \beta} + \frac{g'(z)}{g(z)} \quad (9.7)$$

$$\text{Res}_\beta \frac{f'}{f} = m = \text{the order of the zero } \beta \text{ of } f \quad (9.8)$$

Therefore, $\frac{f'}{f}$ has simple poles at the zeros and poles of f , and

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#(\text{zeros of } f \text{ in Interior}(\gamma)) - \#(\text{poles of } f \text{ in Interior}(\gamma)) \quad (9.9)$$

where the zeros and poles are counted with their multiplicities (i.e. orders). □

9.2.2 Rouché's Theorem

作为辐角原理的推论，下面给出这个强有力的工具 **Rouché's Theorem**.

定理 9.2.2. Rouché's Theorem.

Suppose f, g are holomorphic in an open set containing a contour γ and its interior. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then

f and $f \pm g$ have the same number of zeros in $\text{Interior}(\gamma)$.

证明. 详细证明可见书³P91 Thm 4.3.

□

利用 **Rouché's Theorem**，我们可以证明许多结论，比如代数基本定理 (FTA, §9.3 题 2).

³ 《Complex Analysis》 — Elias M. Stein

9.3 课堂例题 2024 – 04 – 22

1. Determine the number of zeros of

$$p(z) = z^7 + 5z^3 - z - 2 \text{ in } \mathbb{D}$$

解. Let $f(z) = 5z^3$, $g(z) = z^7 - z - 2$. Then $|f(z)| = 5 > 4 \geq |g(z)|$ on $z \in \partial\mathbb{D}$.

Therefore, by **Rouché's Theorem (Thm 9.2.2)**,

$p(z) = f(z) + g(z)$ has the same number of zeros as f in $\mathbb{D}$, which is 3.

□

2. (代数基本定理, FTA)

Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, where $a_1, \dots, a_n \in \mathbb{C}$. Then

$p(z)$ has n zeros in $\mathbb{C}$.

解. Let $f(z) = z^n$, $g(z) = a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$.

If R is sufficiently large, say $R > \max \{1, n \cdot \max_{1 \leq k \leq n} |a_k|\}$, then

$$|f(z)| > |g(z)| \text{ for all } z \in C_R(0)$$

Therefore, by **Rouché's Theorem (Thm 9.2.2)**,

$p(z) = f(z) + g(z)$ has the same number of zeros as f in $\mathbb{C}$, which is n .

□

3. (书 P26 题 7)

Suppose $|a_k| < 1$, $k = 1, 2, \dots, n$.

$$f(z) = \prod_{k=1}^n \frac{z - a_k}{1 - \overline{a_k}z} \quad (9.10)$$

If $|b| < 1$, then $f(z) = b$ has n roots in $\mathbb{D}$.

证明. Let $g(z) = b$. Then $|g(z)| = |b| < |f(z)| = 1$ for all $z \in \partial\mathbb{D}$.

Then by **Rouché's Theorem (Thm 9.2.2)**,

$f(z) - b = f(z) - g(z)$ has the same number of zeros as g in $\mathbb{D}$, which is 0.

□

第十一章 Week 11

11.1 Rouché's Theorem 的更强推论

首先先来回顾一下 **Rouché's Theorem (Thm 9.2.2)**.

定理 11.1.1. Rouché's Theorem.

Suppose f, g are holomorphic in an open set containing a contour γ and its interior. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then

f and $f \pm g$ have the same number of zeros in $\text{Interior}(\gamma)$.

注. 可以容易得到其等价命题为:

- $\Leftrightarrow$ If $|f| > |f - g|$ for all $z \in \gamma$, then

f and g have the same number of zeros in $\text{Interior}(\gamma)$.

- 而事实上, 我们可以给出比 **Rouché's Theorem (Thm 11.1.1)** 更强的结论 (条件减弱).

$\Leftrightarrow$ If $|f| + |g| > |f - g|$ for all $z \in \gamma$, then

f and g have the same number of zeros in $\text{Interior}(\gamma)$.

11.2 Open Mapping Theorem & Maximum Modulus Principle

下面分别给出开映射定理和最大模原理，它们反映了非常值全纯函数的逆天性质.

11.2.1 开映射定理

定理 11.2.1. Open Mapping Theorem.

If f is holomorphic and **nonconstant** in an open set Ω , then f is open.

11.2.2 最大模原理

定理 11.2.2. Maximum Modulus Principle.

If f is a **nonconstant** holomorphic function in a region Ω , then

f cannot attain its maximum in Ω .

注. 条件中的 **holomorphic** 不可减弱为 **continuous**. 反例:

$$g(z) = 1 - |z|^2, z \in \mathbb{D} \text{ attains its maximum at } z = 0.$$

推论 11.2.3. Suppose Ω is a region with compact closure $\overline{\Omega}$. If f is holomorphic in Ω , and continuous on $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |f(z)| = \sup_{z \in \overline{\Omega} - \Omega} |f(z)| \quad (11.1)$$

11.3 *Schwartz Lemma*

下面来介绍 **Schwartz Lemma**, 它是我们后面所要讲的共形映射 (**Conformal Map**) 的基础.

定理 11.3.1. Schwartz Lemma.

Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

If for some $0 \neq z_0 \in \mathbb{D}$, $|f(z_0)| = |z_0|$, then $\exists$ unit complex number c , s. t.

$$f(z) = cz$$

(i.e. $f(z)$ is a rotation)

(ii) $|f'(0)| \leq 1$ and if equality holds, then f is a rotation.

11.4 Exercise 2024-05-06

1. 书 P105 T13-19 , 21-22
2. 书 P109 Problem 2-3

第十二章 Week 12

12.1 复对数

此节我们来回顾一下 §3.1, 3.3 节中有关复对数的有关概念, 并给出一些与之相关的结论.

定义 12.1.1. $\forall z \in \mathbb{C}$, the solution to $e^w = z$ is given by

$$\text{Log}(z) = \log |z| + i(\arg z + 2k\pi), \quad k \in \mathbb{N} \quad (12.1)$$

A holomorphic function $g : U \rightarrow \mathbb{C}$ is called a branch of Logz on U if

$$e^{g(z)} = z \text{ for all } z \in U$$

下面给出一个常用结论及其推论.

命题 12.1.1. Let $\Omega \subset \mathbb{C}$ be simply connected, $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Assume $f(z) \neq 0$, $\forall z \in \Omega$ (**nonvanishing**). Then $\exists$ holomorphic $g : \Omega \rightarrow \mathbb{C}$, s. t.

$$e^{g(z)} = f(z) \quad (12.2)$$

推论 12.1.1. Let $\Omega \subset \mathbb{C}$ be simply connected and let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic and **nonvanishing** on Ω . Then $\exists$ holomorphic $g : \Omega \rightarrow \mathbb{C}$, s. t.

$$(g(z))^2 = f(z) \quad (12.3)$$

12.2 Conformal Mappings

下面进入新的非常有意思的一章内容——共形映射 (Conformal Mappings). 这一章我们将从几何的角度, 研究两个复平面的特定区域之间的几何变换关系.

先来给出共形映射的定义.

定义 12.2.1. A bijective holomorphic function $f : U \rightarrow V$ is called a conformal map or biholomorphism. Given such a mapping f , we say that U & V are conformally equivalent.

注. 即我们将全纯双射叫做共形映射. 事实上, 通过下面的 **Prop 12.2.1**, 可以容易得到共形映射 $f : U \rightarrow V$ 事实上给出了复平面 $\mathbb{C}$ 的所有子区域类上的一个等价关系, 即

$$U \sim V \Leftrightarrow \exists \text{ conformal map } f : U \rightarrow V$$

而我们把这个等价关系称作是共性等价 (conformally equivalent).

下面给出一个强有力的命题, 它给出了全纯单射的逆映射的全纯性.

命题 12.2.1. If $f : U \rightarrow V$ is holomorphic and injective, then

$$f'(z) \neq 0, \forall z \in U$$

Moreover, the inverse of f defined on its range (i.e. $f(U)$) is holomorphic, and thus the inverse of a conformal mapping is holomorphic.

12.3 上半平面 $\mathbb{H}$ 到 $\mathbb{D}$ 的共形映射

先给出一个我们后面常用的记号:

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\} \quad (12.4)$$

下面给出一个上半平面 $\mathbb{H}$ 到 $\mathbb{D}$ 的共形映射.

定理 12.3.1. The map following is a conformal map.

$$F : \mathbb{H} \longrightarrow \mathbb{D} \quad (12.5)$$

$$z \longmapsto \frac{-z + i}{z + i} \quad (12.6)$$

12.4 Exercise 2024-05-13

1. P248 T1.
2. P251 T13.
3. P209 §1.2 例 1 - 8.

第十三章 *Week 13*

13.1 Automorphisms of $\mathbb{D}$, $\mathbb{C}$, $\mathbb{H}$

下面我们来研究复平面上三种特殊区域的自同构群 (Automorphism).

首先给出自同构 (Automorphism) 的概念.

定义 13.1.1. A conformal map from an open set Ω to itself is called an automorphism of Ω .

The set of all automorphisms of Ω is denoted by $Aut(\Omega)$.

下面我们分别对开圆盘 $\mathbb{D}$, 复平面 $\mathbb{C}$, 及上半平面 $\mathbb{H}$ (下一周) 上的自同构群 (Automorphisms) 的结构进行研究和讨论.

13.2 Automorphisms of $\mathbb{D}$

13.2.1 The Blaschke Factors

首先先来介绍一类特殊的，同时也是 $\mathbb{D}$ 上最重要的一类自同构——**Blaschke Factors**:

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z} \in \text{Aut}(\mathbb{D}), \text{ where } a \in \mathbb{C} \text{ with } |a| < 1 \quad (13.1)$$

(可用 **Maximum Modulus Principle (Thm 11.2.2)** 证明 $\varphi_a(\mathbb{D}) \subset \mathbb{D}$)

同时容易得到

$$\varphi_a \circ \varphi_a(z) = z, \quad \forall z \in \mathbb{D} \quad (13.2)$$

13.2.2 Automorphisms of $\mathbb{D}$

在介绍完 **Blaschke Factors** 后，下面我们给出 $\mathbb{D}$ 上自同构的一般的刻画.

定理 13.2.1. Automorphisms of $\mathbb{D}$.

If $f \in \text{Aut}(\mathbb{D})$, then $\exists \theta \in \mathbb{R}, a \in \mathbb{D}$, s. t.

$$f(z) = e^{i\theta} \cdot \varphi_a(z), \text{ where } \varphi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad (13.3)$$

推论 13.2.2. The only automorphism of $\mathbb{D}$ that fixes the origin are the rotations.

13.3 Automorphisms of $\mathbb{C}$

复平面 $\mathbb{C}$ 上的自同构非常的简单，这里要用到书中第三章练习 15 的结论 (习题证明可见我另一篇知乎文章).

定理 13.3.1. Automorphisms of $\mathbb{C}$.

If $f \in \text{Aut}(\Omega)$, then

$$f(z) = az + b \text{ for some } a, b \in \mathbb{C}, a \neq 0 \quad (13.4)$$

13.4 Exercise 2024-05-20

1. P209-212 Examples.
2. P248 T1-7

第十四章 Week 14

14.1 Automorphisms of $\mathbb{H}$

最后一周的内容让我们来研究上半平面 $\mathbb{H}$ 上自同构的结构.

14.1.1 Möbius Transforms / Fractional Linear Transforms (分式线性变换)

先来给出 **Möbius Transform** 的概念.

定义 14.1.1. Given $a, b, c, d \in \mathbb{C}$, s. t. $ad - bc \neq 0$, then

$$z \mapsto \frac{az + b}{cz + d} \text{ is called a } \underline{\text{Möbius Transform}}. \quad (14.1)$$

注. Möbius Transform is a composition of 3 elementary ones:

$$\begin{cases} 1. \text{scaling: } \sigma_p(z) = pz, & p \in \mathbb{C}, p \neq 0 \\ 2. \text{translation: } \tau_q(z) = z + q, & q \in \mathbb{C} \\ 3. \text{inversion: } I(z) = \frac{1}{z} \text{ (反演)} \end{cases} \quad (14.2)$$

下面给出一些记号:

- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) \neq 0$.

$$M_A(z) = \frac{az + b}{cz + d} \quad (14.3)$$

- (实 2 阶特殊线性群)

$$SL_2(\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, \det(A) = 1 \right\} \quad (14.4)$$

关于 **Möbius Transform**, 不难证明有以下事实:

1. **Fact 1:** Let A_1, A_2 be 2 invertible metrics, then

$$M_{A_1 A_2}(z) = M_{A_1} \circ M_{A_2}(z) \quad (14.5)$$

2. **Fact 2:** Möbius Transform preserves lines & circles.

注. 此处指的是

直线 $\rightarrow$ 直线 / 圆

&

圆 $\rightarrow$ 直线 / 圆

3. **Fact 3:** Any circle can be preserved by the equation

$$b|z|^2 - 2\operatorname{Re}(\bar{a}z) + c = 0, \text{ where } a \in \mathbb{C}, b, c \in \mathbb{R}, bc < |a|^2 \quad (14.6)$$

14.1.2 Automorphisms of $\mathbb{H}$

有了以上概念的铺垫，上半平面 $\mathbb{H}$ 上的自同构的描述就显得十分轻松.

定理 14.1.1. Automorphisms of $\mathbb{H}$.

If $f \in \text{Aut}(\mathbb{H})$, then $\exists A \in SL_2(\mathbb{R})$, s. t.

$$f(z) = M_A(z) \tag{14.7}$$

附录 A *L'Hôspital's Rule*

事实上，在复数域 $\mathbb{C}$ 上，**L'Hospital's Rule** 同样成立. 下面便将其推广至 $\mathbb{C}$.

A.1 弱化版本

首先先给出一个常用的弱化版本.

定理 A.1.1. Suppose f, g are holomorphic in a region containing $D_r(z_0)$ for some $r > 0$. If $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \quad (\text{A.1})$$

证明. Since $f(z_0) = g(z_0) = 0$, then

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z \rightarrow z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z \rightarrow z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} = \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} \quad (\text{A.2})$$

□